Constructive Geometry

Commentary on the Constructions

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Introduction

This commentary includes what might be called background material to the constructions. Our class method has been to present each of the constructions as the primary feature, with such background or associative material distinctly secondary and for this reason such material is groped in this separate commentary. For convenience it is organized with specific reference to each of the constructions. The comments to each construction begins with a restatement of the session number and title of the construction. This is followed by an indication of the sources, the full titles of which are listed in the bibliography. These sources include works which are ancient and modern, mathematical, philosophical, architectural and practical. The ultimate source for all the constructions in this class is, of course, Euclid's 'Elements' [Euclid, 350? - 285 BC]. The definitive translation and commentary for our time is that of Heath. He relies mainly on Aristotle and Proclus.

Included here are proofs for those of the constructions which have them. Some of the more specifically architectural constructions do not have proofs, being entirely applications of elements proved elsewhere. Alternate methods of construction to those demonstrated in class are also included for several of the constructions.

Many of the constructions have references in philosophical literature, particularly in Plato [420? - 347 BC]. In this vein also is the late neo-Platonic philosopher Proclus [410 - 485 AD]. His 'Commentary on the First Book of Euclid' is one of the great masterpieces of classical scholarship, looking back over the entire thousand year history of the science of geometry since Pythagoras. Proclus' commentary complements and highlights the brilliance of Euclid's achievement in writing the Elements. For Proclus was heir to the entire thousand year-long classical philosophical tradition, the exoteric lucidity of the Aristotelian mode, in which Euclid writes, and the more esoteric, wholistic mode of Plato and his followers; also to the even older religious tradition of the mysteries, that is, the initiatic systems related to Dionyses, Orpheus, Hermes and many others. Thus in his 'Commentary' Proclus is often contrasting the crystalline lucidity of Euclid with the more opaque mythological references to the ancient polytheistic divinities. This process offers insights, for the classicist, in both directions. It calls attention to the logical achievement of Euclid when the philosophical background is revealed; and it sheds light on the mathematical aspect of myth and even mystery when, for example, Proclus refers to the divinities associated with various geometric concepts. Such Neo-Platonic philosophical ideas connect our 'Constructive Geometry' course with the 'Theory of Proportion' course also given at ICAA by your present author.

In keeping with our practical orientation here, references to various constructions that appear in the canonical literature of architecture are included in the commentary. Leading among these is Sebastian Serlio [1475-1554], whose 'First Book of Architecture' is called 'On Geometry'. It is filled with practical geometrical lore but behind many of his 'homeboy' style problems are some of the more sublime aspects of the subject.

A younger contemporary of Serlio is the much better known Andrea Palladio [1508-1580]. His 'Four Books' found there way to England through the influence of Inigo Jones and Palladio's text later became the basis of the eighteenth century Georgian style. This in turn served as the basis for early American design through the Georgian and Federal periods. Palladio advanced a system of 'harmonic' ratios in design. These small whole number ratios, to be used for the plans of rooms were related to the tonal intervals in the tempered musical octave. For elevation heights Palladio recommended the application of the three principle means - arithmetic, geometric and harmonic -

used in Plato's 'Timaeus' to generate the natural octave. In Palladio we see a transition from the craft based geometrical algebra of Serlio to a more intellectual approach based in classical literature and music.

The canon of Vignola [1507-1573], known as the 'Regola' was influential in France and became incorporated in the Beaux Arts tradition whence it was transposed to our country in the mid-19th century. Vignola also uses a system of harmonic based small whole number ratios to describe the elements of architecture.

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First Principles

A summary of some related and confusing terms used in the following discussion of first principles are briefly defined below:

thesis - a logical idea, assumed as a premise for discussion, which may or may not be provable or related to any observed phenomenon; from the Greek *thesis*, a placing, position …, the base of *tithenai*, to put, place. Several types of thesis are theories, axioms, definitions, hypothesis, and postulates.

theory - a thesis that explains something or establishes causality; in modern terms, usually based on observation. Webster's NWD, College edition, 1960 defines theory *"as coming from the Greek* theoria*, a looking at, contemplation, speculation;* theorein*, to look at.* Theory*, implies considerable evidence in support of a formulated general principle explaining the operation of certain phenomena;* hypothesis *implies an inadequacy of evidence in support of an explanation that is tentatively inferred, often, as a basis for further experimentation;* law *implies an exact formulation of the principle operating in a sequence of events in nature observed to occur with* *unvarying uniformity under the same conditions."* A theory is not a fact or a law. It is a connecting of the dots [data points] which leads to a hypothesis, the asking of yes or no questions.

axiom or **common notion** - a thesis which the pupil must know to learn anything - Axioms are 'not demonstrable of proof' - but are 'primary premises of demonstration' [Aristotle PA I.10, 76b, 14] and 'contribute to proof' [Proclus 209].

definition - a statement of attributes in words which says nothing as to the existence of its subject.

hypothesis - a thesis 'made with the assent of the learner' [Aristotle PA I.6-10]; assumes one side or another - a hypothesis 'assumes the existence or non existence of defined entities' [Aristotle PA I.2, 72a, 14-24] In modern terms, a question asked on the basis of a theory, a question that has a yes or no answer, that can be tested.

postulate - a thesis made without the assent of the learner [Aristotle PA I.6-10], 'contributory to constructions' [Proclus 209]

First principles are not demonstrable. This should be of note to modern, rationalistic readers. There is a tendency today for people to endow logic with exclusive access to truth. But in the Timaeus [29c]we are told, *"If in our treatment of a great host of matters regarding the Gods and the generation of the Universe we prove unable to give accounts that are always in all respects self-consistent and perfectly exact, be not thou surprised; rather we should be content if we can furnish accounts that are inferior to none in likelihood, remembering that both I who speak and you who judge are but human creatures, so that it becomes us to accept the likely account of these matters and forbear to search beyond it"*. All rational based study, such as geometry, must start from non-provable, that is to say, trans-rational premises. Thus all logical study, including modern science, does not provide direct access to truth but are 'likely stories'.

And we get this requirement not only from Plato but from Aristotle as well. Heath, summarizing Aristotle [PA I.6-10], writes [p 119], *"Every demonstrative science must start from indemonstrable principles: otherwise, the steps of demonstration would be endless. Of these indemonstrable principles some are (a) common to all sciences, others are (b) particular, or peculiar to the particular science; (a) the common principles are the 'axioms', most commonly illustrated by the axiom that, if equals be subtracted from equals, the remainders are equal. Coming now to (b) the principles peculiar to the particular science which must be assumed, we have first the 'genus' or subject matter, the 'existence' of which must be assumed, viz., magnitude in the case of geometry, the unit in the case of arithmetic. Under this we must assume 'definitions' of manifestations or attributes of the subject matter, e.g. straight lines, triangles, etc. the definition in itself says nothing as to the existence of the thing defined: it only requires to be understood. But in geometry, in addition to the 'genus' and the 'definitions', we have to assume the 'existence' of a few 'primary' things which are defined, viz. points and lines only: the existence of everything else, e.g. the various figures made up of these ... and their properties ... has to be proved ...by construction and demonstration. ..."*

"We have then clearly distinguished, among the indemonstrable principles, 'axioms' and 'definitions'. A 'postulate' is also distinguished from a 'hypothesis', the latter being made with the assent of the learner, the former without such assent or even in opposition to his opinion ... ."

Heath continues [p 120], *"Aristotle's distinction also between 'hypothesis' and 'definition', and between 'hypothesis' and 'axiom', is clear from the following passage* [Posterior Analytics I.2, 72a, 14-24]: *'...I call* [a basic truth of logical reasoning] *a 'thesis' that which it is neither possible to prove nor essential for any one to hold who is to learn anything; but that which it is necessary for any one to hold who is to learn anything whatever is an 'axiom' ... But, of 'thesis', one kind is that which assumes one or other side of a predication, as, for instance, that something exists or does not exist, and this is a 'hypothesis; the other, which makes no such assumption, is a 'definition'. For a* *definition is a thesis: thus the arithmetician posits that a unit is that which is indivisible in respect of quantity; but this is not a hypothesis, since what is meant by a unit and the fact that a unit exists are different things."*

Proclus [75] writes, *"No science demonstrates its own first principles or presents a reason for them; rather each holds them as self-evident, that is, as more evident than their consequences.* [Thus] *it was incumbent on Euclid to set apart the principles from their consequences; and this is just what he does ... besides setting forth at the outset of his whole treatise the common principles of the science."*

[76] *"Next he divides them* [the principles] *into hypotheses, postulates, and axioms ... When a proposition that is to be accepted into the rank of first principles is something both known to the learner and credible in itself, such a proposition is an axiom: for example, that things equal to the same thing are equal to each other. When the student does not have a self-evident notion of the assertion proposed but nevertheless posits it and thus concedes the point to his teacher, such an assertion is a hypothesis. That a circle is a figure of such-and-such a sort we do not know by a common notion in advance of being taught, but upon hearing it we accept it without a demonstration. Whenever, on the other hand, the statement is unknown and nevertheless is taken as true without the student's conceding it, then, he says, we call it a postulate: for example, that all right angles are equal. ... In this way axiom, postulate, and hypothesis are distinguished according to Aristotle's teaching. Often, however, they are all called hypotheses, just as the Stoics call every simple statement an axiom, so that according to them even hypotheses are axioms, whereas according to others axioms are hypotheses."*

The key to keep in mind is that all rational studies, from nuclear physics to evolution, are based on trans-rational principles or ideas. Such principles or ideas are rooted in Nous and are accessed through the human imagination. Such rational studies offer 'likely stories, which may yield power over matter, but such power is not to be confused with 'truth', which is metaphysical. Geometry, one of the seven Liberal Arts, begins with trans-rational axiomatic assumptions, such as points, lines and planes, then proceds by use of logical principles. Because geometry requires contemplation of ideas it exercises the imagination and can contribute to what Plato calls 'clarifying the orbits of the soul'.

Methods

Terms used in this discussion are:

propositions - an idea put forth requiring demonstration or proof before acceptance. Two kinds of propositions are:

problems - concerned with establishing or demonstrating the existence of something.

theorems - concerned with demonstrating an inherent property of something.

Proclus writes [200-01], *"Science as a whole has two parts: in one it occupies itself with immediate premises, while in the other it treats systematically the things that can be demonstrated or constructed from these first principles, or in general are consequences of them. This second part, in geometry, is divided into the working out of problems and the discovery of theorems. It calls 'problems' those propositions whose aim is to produce, bring into view, or construct what in a sense does not exist, and 'theorems' those whose purpose is to see, identify, and demonstrate the existence or nonexistence of an attribute. Problems require us to construct a figure, or set it at a place, or apply it to another, ... ; theorems endeavor to grasp firmly and bind fast by demonstration the attributes and inherent properties belonging to the objects that are the subject-matter of geometry."*

Proclus [81] adds, *"That Euclid's Elements contain both theorems and problems will be evident from his practice of placing at the end of his demonstrations sometimes 'This is what was to* *be done'* [for a problem, quod erat faciendum, abbreviated QEF] *and at other times 'this is what was to be proved'* [a theorem, QED].*"*

The propositions in Euclid have a particular structure described by Proclus [203]; *"Every problem and every theorem that is furnished with all its parts should contain the following elements: an enunciation, an exposition, a specification, a construction, a proof, and a conclusion. Of these the enunciation states what is given and what is being sought from it. ... The exposition takes separately what is given and prepares it in advance for use in the investigation. The specification takes separately the thing that is sought and makes clear precisely what it is. The construction adds what is lacking in the given for finding what is sought. The proof draws the proposed inference by reasoning scientifically from the propositions that have been admitted. The conclusion reverts to the enunciation, confirming what has been proved."*

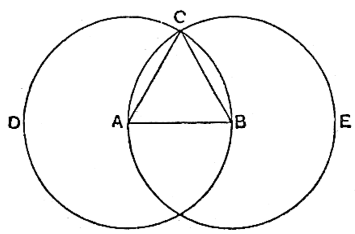
Proclus [208-10] elaborates this format using the example of Euclid's proposition I.1 -'On a given finite straight line to construct an equilateral triangle', similar to our construction 1.6, shown in **figure 1**; *"Clearly it is a problem, for it bids us devise a way of constructing an equilateral triangle. In this case the enunciation consists of both what is given and what is sought. What is given is a finite straight line, and what is sought is how to construct an equilateral triangle on it. The statement of the given precedes and the statement of what is sought follows, so that we may weave them together as 'If there is a finite straight line, it is possible to construct an equilateral triangle on it.' If there were no straight line, no triangle could be produced, for a triangle is bounded by straight lines; nor could it if the line were not finite, for an angle can be constructed only at a definite point, and an unbounded line has no end point."*

Euclid, Book I, Proposition 1

On a given finite straight line to construct an equilateral triangle.

Let *AB* be the given finite straight line.

Thus it is required to construct an equilateral triangle on the straight line *AB*.



With centre *A* and distance *AB* let the circle *BCD* be described; [Post. 3]

again, with centre *B* and distance *BA* let the circle *ACE* be described; [Post. 3]

and from the point *C*, in which the circles cut one another, to the points *A*, *B* let the straight lines *CA*, *CB* be joined. [Post. 1]

Now, since the point *A* is the centre of the circle *CDB*,

AC is equal to *AB*. [Def. 15]

Again, since the point *B* is the centre of the circle *CAE*,

BC is equal to *BA*. [Def. 15]

But *CA* was also proved equal to *AB*;

therefore each of the straight lines *CA*, *CB* is equal to *AB*.

And things which are equal to the same thing are also equal to one another; [*C. N.* 1]

therefore *CA* is also equal to *CB*.

Therefore the three straight lines *CA*, *AB*, *BC* are equal to one another.

Therefore the triangle *ABC* is equilateral; and it has been constructed on the given finite straight line *AB*.

(Being) what it was required to do.

Figure 1

"Next after enunciation is the exposition: 'Let this be the given finite line'. You see that the exposition itself mentions only the given, without reference to what is sought. Upon this follows the specification: 'It is required to construct a equilateral triangle on the designated finite straight line'. In a sense the purpose of the specification is to fix our attention; it makes us more attentive to the proof by announcing what is to be proved, just as the exposition puts us in a better position for learning by producing the given element before our eyes. After the specification comes the construction: 'Let the circle be described with center at one extremity of the line and the remainder of the line as distance; again let a circle be described with the other extremity as center and the

same distance as before; and then from the point of intersection of the circles let straight lines be joined to the two extremities of the given straight line.' You observe that for the construction I make use of the two postulates that a straight line may be drawn from any point to any other and that a circle may be described with [any] *center and distance. In general the postulates are contributory to constructions and the axioms to proofs."*

"Next comes the proof: 'Since one of the two points on the given straight line is the center of the circle enclosing it, the line drawn to the point of intersection is equal to the given straight line. For the same reason, since the other point on the given straight line is itself the center of the circle enclosing it, the line drawn from it to the point of intersection is equal to the given straight line.' These inferences are suggested to us by the definition of the circle, which says that all the lines drawn from its center are equal. 'Each of these lines is therefore equal to the same line; and things equal to the same thing are equal to each other' by the first axiom. 'The three lines therefore are equal, and an equilateral triangle [ABC] has been constructed on this given straight line.' ..."

"To these propositions he adds: 'This is what it was required to do', thus showing that this is the conclusion of a problem ...

Proclus concludes, *"We have thus exercised ourselves and clarified all these distinctions by applying them to a single case, the first problem. The student should do this also for the remaining propositions, asking which of the principle elements are included and which are left out, in how many ways the given is formulated, and what are the principles from which we obtain the construction or the proof. For a comprehensive survey of these matters will provide no little exercise and practice in geometrical reasoning."*

Heath goes on at some length with Proclus and other geometers on the exquisite gradations of the structure of Euclid's propositions. For our purposes here we may note that all the propositions in this course are problems. None are theorems. Nor do we engage in presentation or discussion of first principles except for a minimum number of definitions required to establish the possibility of performing our constructions and to avoid confusion.

The format of our class presentation also begins with an enunciation. In our sessions the task is always to construct something and the statement of that which is required precedes that which is given. Our exposition is usually stated in the first numbered item, which restates the given. We do not then give the specification but move directly to the construction, individually enumerating the necessary steps, as this is our major object. We do not include the proof in the presentation of the constructions but have included selected proofs here in this commentary. All other matters pertaining to the constructions are also consigned to this commentary. The last sentence of the last numbered item in our constructions is the conclusion and restates the first part of our enunciation; thus, as Proclus [210] says, *" ... joining the end to the beginning in imitation of the Nous that unfolds itself and then returns to its starting point."*

Definitions

Point Euclid, I.Def. 1. *'A point is that which has no parts.'* Serlio, p 7, Def. 1. *'A point is an indivisible thing which has in itself no dimensions.'* The point is not defined in Palladio or Vignola. Benjamin pl I, *'A point is that which has position, but no magnitude nor dimensions; neither length, breadth, nor thickness.'*

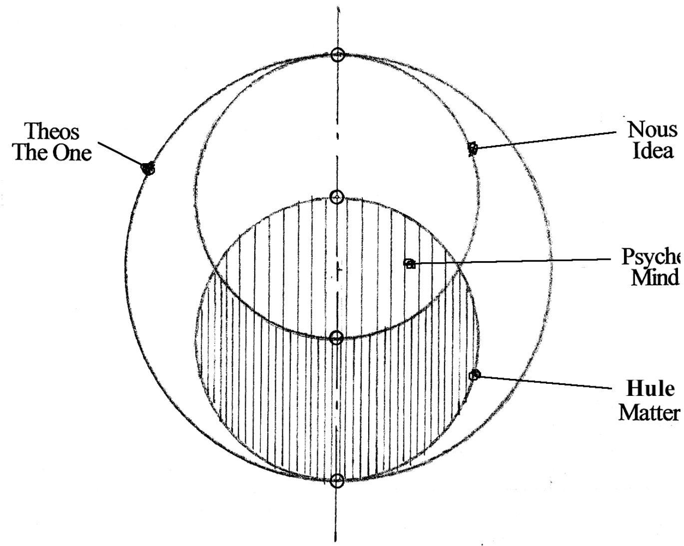
The first principle of geometry is the point. The point is related to the Pythagorean concept of the Monad, the initial and indivisible oneness. The point's existence, for the purpose of this study, is presumed, not demonstrated. Thus the point is given, as we might say, by definition. Proclus says [94], *'By denying parts to it, Euclid signifies to us that the point is the first principle of the entire subject under examination'.*

If the point has no parts it must also be without dimension, that is, unextended dimensionally, or having no magnitude. It is without dimension because any dimension would make it divisible.

Thus the point may also be defined by negation. Heath, for example, [vol. I, p 155] writes of a pre-Euclidean definition, *"Aristotle (Topics VI 4, 141 b 20), in speaking of '<u>the</u> definitions' of point, line, and surface, says that they all define the prior by means of the posterior, a point as an extremity of a line, a line of a surface, and a surface of a solid."*

It may fairly be asked here, what kind of entity has no parts or magnitude? Where could such an entity exist? How could such an entity become anything? To approach such questions in the spirit of seeking a 'most likely story' we may look to the masterful commentary of the fifth century AD Neo-Platonic philosopher Proclus.

The Proclean cosmos, called Theos or the One, may be divided into three overlapping parts or aspects known as nous, psyche and hule, **figure 2**. Nous means intelligence or idea. This realm is the home of the 'eide', the forms or archetypes which are completely disembodied and are outside of time. The opposite end of the spectrum is unformed matter called 'hule'. By itself, hule has no 'form', thus is in a state of 'chaos'. Neither of these states is sensible in that the eide are not embodied and the hule has no identifiable form. Perception can only take place in a realm which overlaps or mediates between the opposites. This is known as 'psyche', or soul, in the ancient sense; or, in a more modern sense, psyche may be translated as mind. Within psyche the eide are projected or impressed, as it were, into matter thus creating the appearance of an orderly cosmos. The eide enter the human mind in the aspect of the imagination. This is a zone where the eide of the Monad, the indivisible unit, transits into the point, becomming the monad that has position, the indivisible beginning of geometry.



The Neo-Platonic Cosmos of Proclus

Figure 2

On the imagination Proclus [52] writes, *"For imagination, both by virtue of its formative activity and because it has existence with and in the body, always produces individual pictures that have divisible extension and shape, and everything that it knows has this kind of existence. For this reason* [Aristotle, De Anima 430a24] *has ventured to call it 'passive Nous'. ... he intended ... to express the middle position it occupies between the highest and the lowest types of knowledge and so called it at the same time 'nous' because it resembles the highest, and 'passive' because of its kinship with the lowest. For the knowing which is not of shapes and figures has its intelligible objects in itself, and its activity is concerned with these, its own contents; it is itself one with the things it knows, free of any impression or affection coming from elsewhere. But the lowest forms of knowledge work through the sense organs; they are more like affections, receiving their opinions from without and changing as their objects change. Such is what sense-perception is, the result of 'violent affections', as Plato says* [Timaeus 42a]."

Proclus continues [52-53], *"By contrast the imagination, occupying the central position in the scale of knowing, is moved by itself to put forth what it knows, but because it is not outside the body, when it draws its objects out of the undivided center of its life, it expresses them in the medium of division, extension, and figure, For this reason everything that it thinks is a picture or a shape of its thought. It thinks the circle as extended, and although this circle is free of external matter, it possesses an intelligible matter provided by the imagination itself. This is why there is more than* *one circle in the imagination, as there is more than one circle in the sense world; for with extension there appear also differences in size and number among circles and triangles."*

With this in mind we may turn to what Heath [vol. I p 155] notes as the earliest definition of the point: "a monad having position". Proclus [95-96] writes, *'Since the Pythagoreans define the point as a unit that has position, we ought to inquire what they mean by saying this. That numbers are purer and more immaterial than magnitudes and that the starting-point of numbers is simpler than that of magnitudes are clear to everyone. But when they speak of the unit as not having position, I think they are indicating that unity and number - that is, abstract number - have their existence in thought; and that is why each number, such as five or seven, appears to every mind as one and not many, and as free of any extraneous figure or form. By contrast the point is projected in imagination and comes to be, as it were, in a place and embodied in intelligible matter. Hence the unit is without position, since it immaterial and outside all extension and place; but the point has position because it occurs in the bosom of imagination and is therefore enmattered. Owing to its affinity with the principles, the unit is simpler than the point; for the point, by having position, goes beyond the unit. And additional determinants in the bodiless concepts effect a lessening of being in the things that accept them."*

We may thus say that the Monad, the unit of Proclus, exists in the realm of Nous, the intelligible; while the point exists within the imagination, within psyche. When extended into the material realm the point becomes the 'dot'. This distinction may be related to Heath's interesting note [156] that an early term for the point was 'stigma', implying a puncture or void; while the term used in Euclid and latter is 'sameion', or nota, a mark. The student should thus keep in mind that the dot is a material representation of the point which in turn is a projection in the imagination of the Monad; and that no human being has ever seen a point, a line or plane, but only their material representations.

Geometry as well as the other ancient sciences of number, arithmetic, music and astronomy may thus be seen as originating in the imagination or at least requiring the exercise of that faculty of psyche. And because psyche is intermediate between embodied matter and bodiless eide such studies may assist us to lift our mental horizons upward, to liberate ourselves, so to speak, from entrapment in matter. The activity of design, we may note, also originates in the imagination. Thus there is a profound affinity of geometry and design. Both may be utilized for the work of self elevation and intellectual clarification.

Line A line has length but no other dimensions.

Euclid I, Def. 2 A line is breathless length.

Proclus begins [97], *"The line is second in order as the first and simplest extension, what our geomete*r [Euclid] *calls 'length', adding 'without breadth' because the line also has the relation of a principle to the surface, He taught us what the point is, through negations only, since it is the principle of all magnitudes; but the line he explains partly by affirmation and partly by negation. The line is length, and in this respect it goes beyond the undivideness of the point; yet it is without breadth, since it is devoid of the other dimensions. For everything that is without breadth is also without depth ..."*

"The line has also been defined in other ways. Some [Aristotle, De. An. 409a4, 'a moving line generates a surface and a moving point a line…'] *define it as the 'flowing of a point', others as 'magnitude extended in one direction'. The latter definition indicates perfectly the nature of the line, but that which calls it the flowing of a point appears to explain it in terms of its generative cause and sets before us not line in general, but the material line. This line owes its being to the point, which, though without parts, is the cause of the existence of all divisible things; and the 'flowing' indicated*

the forthgoing of the point and its generative power that extends to every dimension without diminution and, remaining itself the same, provides existence to all divisible things."

[97-98] *"... let us recall the more Pythagorean doctrine that posits the point as analogous to the monad, the line to the dyad, the surface to the triad, and the solids to the tetrad. On the other hand, considering them as extended, we shall find that the line is one-dimensional, the surface two-dimensional, and the solid three-dimensional; hence Aristotle says [De Ca. 268a8] that body comes to completion with the number three. ... Both of these orderings have their justification, but that of the Pythagoreans is closer to first principles, for it starts from the top and follows the nature of things. The point is twofold, because it exists either by itself or in the line. As a limit ... it is a likeness of the very summit of being and so is ranked as analogous to unity. ... Since the line is the first thing to have parts and to be a whole, and since it is both monadic because unidimensional and dyadic because of its forthgoing ... for these reasons it is an imitation of wholeness and of that grade of being which is extended oneness and generated duality. For this it is that produces transformation into length, that is, into divisible extendedness in one dimension together with participation in duality. The surface is both triad and dyad; being the receptacle of the primary figures as well as the first nature that takes on form and shape, it resembles both the triad that primarily bounds all beings and also in a way the dyad which divides this triadic nature. But the solid, extended in three directions and defined by the tetrad that comprehends all ration in itself, carries our thoughts to that intelligible cosmos which by the aid of the tetradic property - that is, the feminine and generative power - produces the separation of the orders of bodily things and the division of the universe into three."*

Proclus [100] closes with these observations, *"...we should also accept what the followers of Apollonius say, namely, that we have the idea of the line when we ask only for a measurement of length, as of a road or a wall. For breadth does not enter into our consideration ... And we can get a visual perception of the line if we look at the middle division separating lighted from shaded areas, whether on the moon or on the earth. For the part that lies between them is unextended in breadth, but it has length, since it is stretched out all along the light and the shadow."*

Euclid I, Def. 3 The extremities of a line are points.

Proclus [101-2] makes the distinction here between unbounded lines, such as the circle, bounded lines, such as an interval between any two points, and lines with one boundary, such as when two lines, unbounded in one direction, both begin at a point, forming an angle; Heath, paraphrasing Proclus writes, *"if a line has extremities, those extremities are points."*

Heath continues [p 165], *"It being unscientific, as Aristotle said, to define a point as the 'extremity of a line', thereby explaining the prior by the posterior, Euclid defined a point differently; then, as it was necessary to connect a point with a line, he introduced this explanation after the definitions of both had been given. This compromise is no doubt his own idea ..."*

"We miss a statement of the facts, equally requiring to be known, that a 'division' of a line, no less than its 'beginning' or 'end', is a point [Aristotle, Met. 1060b15], and that the intersection of two lines is also a point." If Euclid had added this, Heath says, *"Proclus would have been spared the difficulty which he finds in the fact that some of the lines used in Euclid (namely the infinite straight lines on the one hand and circles on the other) have no 'extremities'."* Apparently, all three commentators seem to have found a chink, albeit a small one, in Euclid's highly polished logical armor.

Straight line A straight line is the shortest distance between two points. Or it may be said that any part of it fits on any other part regardless of orientation.

Euclid I, Def. 4 A straight line is a line which lies evenly with the points on itself.

As usual, Proclus [104] begins his discussion with a cosmological reference, *"Plato assumes that the two simplest and most fundamental species of line are the straight and the circular and makes all other lines mixtures of these two … Since there are three hypostases below the One - namely, the Limit, the Unlimited, and the Mixed - it is through them that the species of lines, angles, and figures come to be. Corresponding to the Limit are, in surfaces, … the circle … and in solids, the sphere. To the Unlimited corresponds the straight line in these three groups." Examples of mixed figures are spiral lines, semicircular figures and in solids, cones and cylinders. …* [107] *It appears that the circular line belongs with the Limit and has that relation to other lines that the Limit has to all things; for of the simple lines the circular line is limited and makes a figure, whereas the straight line belongs with the Unlimited and hence can be projected indefinitely without end. So as all other things arise from the Limit and the Unlimited, likewise the whole class of mixed lines … come from the circle and the straight line."*

[108] *"For this reason the soul contains in advance the straight and the circular in her essential nature, so that she may supervise the whole array of unlimiteds as well as all the limited beings in the cosmos, providing for their forthgoing by the straight line and for their reversion by the circle, leading them to plurality by the one and collecting them all into unity by the other. And not only the soul, but also he who constituted the soul and furnished her with these two powers possesses in himself the primordial causes. For 'holding in advance the beginning, middles, and ends of all things', says Plato, 'he bounds straight lines as he moves around by nature', He goes forth to all things with his providential activity while he is turned upon himself 'abiding in his accustomed nature', as Timaeus says. … The demiurgic Nous has therefore set up these two principles in himself, the straight and the circular, and produced out of himself two monads, the one acting in a circular fashion to perfect all intelligible essences, the other moving in straight line to bring all perceptible things to birth. Since the soul is intermediate between sensibles and intelligibles, she moves in circular fashion insofar as she is allied to intelligible Nous but, insofar as she presided over sensibles, exercises her providence in a straight line."*

Proclus [109] returns to Euclid's definition, *"Euclid gives the definition of the straight line that we have set forth above, making clear by it that the straight line alone covers a distance equal to that between the points that lie on it. For the interval between any two points is the length of the line that these points define, and this is what is meant by 'lying evenly with the points on itself'. If you take two points on a circle or any other kind of line, the length of the line between the two points taken is greater than the distance between them. Hence it accords with a common notion that those who go in a straight line travel only the distance they need to cover …"*

"Plato, however, defines the straight line as that whose middle intercepts the view of the extremes. This is a necessary property of things lying on a straight line but need not be true of things on a circle or any other extension. …"

[110] *"But Archimedes defined the straight line as the shortest of all lines having the same extremities. Because, as Euclid's definition says, it lies evenly with the points on itself, it is the shortest of all lines having the same extremities; for if there were a shorter line, this one would not lie evenly with its own extremities. In fact all other definitions of the straight line fall back upon the same notion - that it is a line stretched to the utmost, that one part of it does not lie in a lower and another in a higher plane, that all of its parts coincide similarly with all others, that it is a line that remains fixed if its end points remain fixed, that it cannot make a figure with another line of the same nature. All these definitions express the property which the straight line has by virtue of being simple and exhibiting the single shortest route from one extremity to the other," To which Morrow, Proclus' translator adds, [note 110.23] "The first, third and fourth of these alternative 'definitions' are found in Heron, and the second and fifth are found in Euclid himself (I.4; XI.1), though not as definitions (Heath, Euclid I, 168)."*

Heath begins [p 165], *"The only definition of a straight line authenticated as pre-Euclidean is that of Plato, who defined it as 'that of which the middle covers the ends' (relatively, that is, to an eye placed at either end and looking along the straight line). It appears in the Parmenides 137 E: 'straight is whatever has its middle in front of (i.e. so placed as to obstruct the view of) both its ends'. Aristotle quotes it in equivalent terms (Topics VI, II, 148b27), and, as he does not mention its author, but states it in combination with the definition of a line as the extremity of a surface, we may assume that he used it as being well known. Proclus also quotes the definition as Plato's in almost identical terms (109.21). This definition is ingenious, but implicitly appeals to the sense of sight and involves the postulate that the line of sight is straight (cf. the Aristotelian Problems 31, 20, 959a39 …). As regards the straightness of 'visual rays', cf. Euclid's own Optics, Def. I, 2 assumed as hypotheses, in which he speaks of the 'straight lines' drawn from the eye, avoiding the word 'rays', and then says that the figure contained by the 'visual rays' is a cone with its vertex in the eye. As Aristotle mentions no definition of a straight line resembling Euclid's but gives only Plato's definition and the other explaining it as the 'extremity of a surface', the latter being evidently the current definition in contemporary textbooks, we may safely infer that Euclid's definition was a new departure of his own."*

Heath [p 166] now goes on to critique Proclus, *"Thus Proclus would interpret somewhat in this way: 'a straight line is that which represents extension equal with (the distance separating) the points on it'. This explanation seems to be an attempt to graft on to Euclid's definition the* <u>*assumption*</u> *[emphasis by Heath] of Archimedes (On the sphere and cylinder I. ad init.) that 'of all the lines which have the same extremities the straight line is least'. …"* According to Heath [167], Proclus tries to make the same words, 'ex isou', mean 'even with' and 'at a distance'. Heath goes on to try to elucidate the Greek terms involved, coming to the conclusion that the language is 'hopelessly obscure'

Heath [168] continues, *"the question arises, what was the origin of Euclid's definition, or how was it suggested to him? It seems to me that the basis of it was really Plato's definition of a straight line as 'that line the middle of which covers the ends'. Euclid was a Platonist, and what more natural than that he should have adopted Plato's definition in substance, while regarding it as essential to change the form of words in order to make it independent of any implied appeal to vision, which, as a physical fact, should not properly find a place in a purely geometrical definition."*

Heath now makes this remarkable statement, *"The truth is that Euclid was attempting the impossible. As Pfleiderer says (Scholia to Euclid), 'It seems as though the notion of a 'straight line', owing to its simplicity, cannot be explained by any regular definition which does not introduce words already containing in themselves, by implication, the notion to be defined (such e.g. are direction, equality, uniformity or evenness of position, unswerving course), and as though it were impossible, if a person does not already know what the term 'straight' here means, to teach it to him unless by putting before him in some way a picture or a drawing of it'."* After reviewing the five definitions given above by Proclus and several more modern ones Heath [p 169] concludes, *"the above definitions all illustrate the observation of Unger (Die Geometric des Euklid, 1833), 'Straight is a simple notion, and hence all definitions of it must fail' …"*

Session 1

We begin the course with several basic constructions utilizing the properties of lines.

1.1 Bisect a line 3

Euclid 1.10. Not in Serlio. Benjamin pl. II, fig. 5. Greenberg p 7. Not in French.

This construction uses the Vesica Pisces, defined as two circles related such that the center of one lies on the circumference of the other. In the Pythagorean tradition the Vesica Pisces is considered as an area of the merger of Idea and Matter, that is to say, the realm of Psyche or Mind,

figure 2. It is in this realm that Mind, in its imaginative capacity, gives birth, so to speak, to sensible form.

Let's look at some of the remarkable properties of this figure. Lines AB and CD meet at right angles and bisect each other. Furthermore, ABC and ABD are equilateral triangles. Thus it is the basis for figures 1.2, 1.3, 1.4, 1.6, 1.7, 2.1, 2.4, 2.5, 3.1, 3.2, and 3.3. Furthermore if the radius AB is taken as an interval of 1 then CD is equal to the $\sqrt{3}$. In these ways the Vesica Pisces becomes a mechanism, a tool for actualizing primary operations, like bisection, and primary shapes like the equilateral triangle.

The Vesica Pisces, in these constructions involving line, is a special case of vesicas in general; that is, the figure formed from the intersection of any two arcs of equal radius. Our use of the Vesica Pisces is partly to indicate its close relation to unity and partly in homage to Euclid, for whom it is the method of Proposition 1.1. Our opening sequence is also Greenberg's.

1.2 Construct a perpendicular from a point on a line. 4

Euclid 1.11. Not in Serlio, given by definition, p 7. Benjamin pl. II. fig. 1, 2 and 3. Greenberg p 7. French, 5.6, p 63.

Any point on a line can be treated as the center of a vesica Pisces. In this case we construct a vesica in reverse. Note that DEF is an equilateral triangle.

1.3 Construct a perpendicular from a point off a line. 5

Euclid 1.12. Not in Serlio. Benjamin pl. II. fig. 4. French, 5.5, p 63.

Also a reverse Vesica Pisces construction.

1.4 Bisect an angle. 6

Euclid 1.9. Not in Serlio. Benjamin pl. II. fig. 8. Greenberg, p 7. Not in French.

This construction is also based in the method of the vesica. If interval DE is used as the radius of the arcs used to find point F, we see the nature of the relationship to the Vesica Pisces.

1.5 Construct a line parallel to a given line. 7

Euclid 1.31. Not in Serlio, given as a definition, p 7. Not in Benjamin, given as a definition. French, 5.3 and 5.4

Our title should include the phrase 'through a given point off the line'. To construct a straight line parallel to a given straight line at any distance one would set any convenient radius and strike arcs through A and B and draw the parallel straight line tangent to the two arcs.

Our construction 1.5 works because of the properties of similar triangles CDE and CDF. Because all three sides are equal the opposite angles FDC and CDE must be equal and the lines AB and FC must be parallel. Euclid has established this in propositions I.27, 28, and 29 and in a series of earlier propositions dealing with the properties of triangles. I.4 thru I.8.

Here begins a series of polygon constructions continuing thru 3.5.

1.6 Construct an equilateral triangle given a side. 8

Euclid I.1; see figure 1 above. Not in Serlio, given as a definition, p 7; the Vesica Pisces is also given by definition p 8. Benjamin pl. II. fig. 11. Greenberg p 14.

This is the 'classic' Vesica Pisces construction. The equilateral triangle may be said to be the first polygon, a figure whose sides and interior angles are equal. The equilateral triangle may be said to be a perfect threeness of three; three sides, three equal intervals, and three equal angles. In this context it is not surprising to find it first among Euclid's propositions.

1.7 Construct an equilateral triangle given the altitude. 9

Not given in Euclid. Not in Serlio. Not in Benjamin. Greenberg p13.

The altitude of the equilateral triangle equals √3/2 = .866… .

Session 2

This session treats the four and five sided polygons and the Golden Section.

2.1 Construct a square given a side. 10

Euclid I.46. Not in Serlio, given by definition, p 8. Benjamin pl. II. fig. 12. Not in French. Greenberg p 21.

We use a combination of two methods, starting with Euclid's raising a perpendicular and laying off the square's side but where he then utilizes the properties of parallel lines to establish the square the method used by Greenberg utilizes the properties of similar triangles, AFB and CBF. The figure made by the two quarter-circle arcs with centers at F and B, connecting points A and G, may be referred to as a 'square vesica'. This figure is used by Serlio [p 24-25] in his remarkable geometrical method for designing urns and vessels.

2.2 Construct a square given a diagonal 11

Euclid 4.6. Not in Serlio. Not in Benjamin. Not in French. Greenberg p 21.

Euclid begins with the circle and then draws two diameters at right angles, proving the square by the properties of triangles in circles, one of the major themes of his Book III, and by the properties of similar triangles. We follow Greenberg who starts with the diagonal of the square, using it as the diameter of the circle, a more workman-like method. This construction is the basis for Vignola's division of the eye of the Ionic volute, shown in our construction 6.6.

The doubling of this square is used in the Meno dialog [84c-d] to establish Plato's theory of internal knowledge and a varient is used as an astrological chart

2.3 Construct a rectangle whose sides are in the Golden Ratio. 12

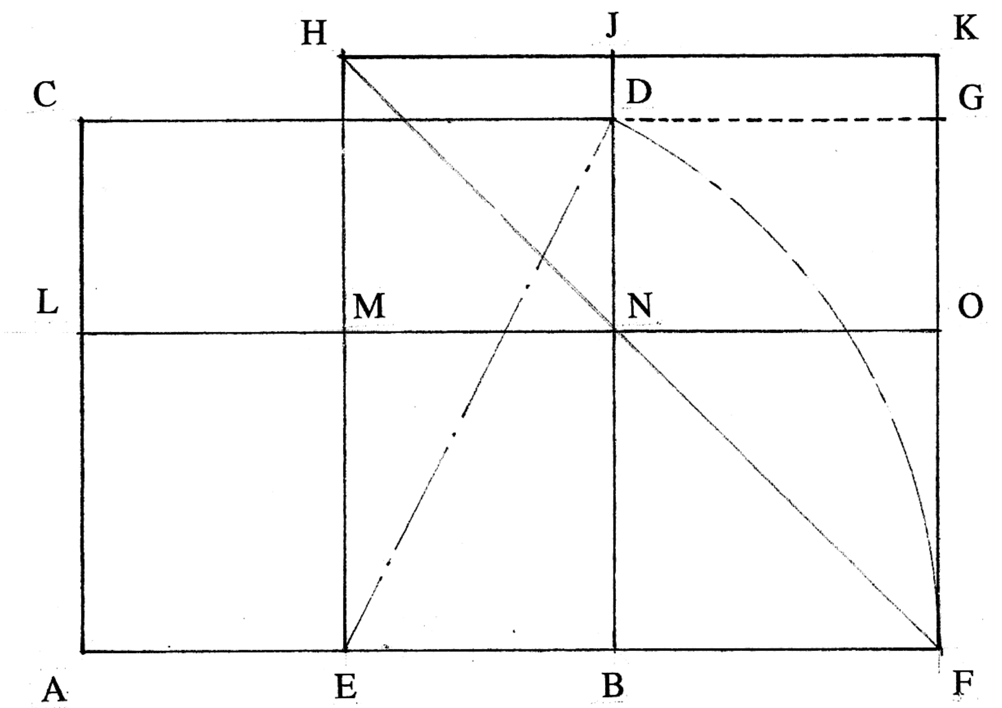
Euclid II.11; see **figure 3**. Not in Serlio, but he does give the 3 x 5 rectangle as one of a family of seven on page 30. Not in Benjamin. Not in French. Greenberg p17.

The theme of Euclid's book II is the transformation of areas, a study also known as 'geometrical algebra'. The general goal of II.11 is to transform a square into a rectangle of equivalent area. Specifically to find a point in that transformation at which the extension beyond the square equals the area taken from the square, and the shape of that extension is also a square. The proposition is put this way: 'To divide a line into two unequal segments such that the square of the larger is equal in area to the rectangle whose sides are the whole line and the smaller segment'. This point will be the 'extreme and mean ratio' also known as the Golden Section or phi, Ø. Thus we have a geometrical definition of Ø. It is the unique place where the transformed area takes a shape analogous to the original area; where the area taken from a square is also a square; the point at which an image of the original unity is obtained through a process of division.

Euclid limits the goal of II.11 to proving that the area of square BNOF equals rectangle LCDN. He begins by giving line AB and setting the extension BF by bisecting AB at E and making EF equal to ED and this is what we have done in our construction 2.3. Euclid invokes II.6 to establish that rectangle ALME = NJKO and that rectangle ALOF + square MHJN = square EHKF. Since line ED = EF he uses I.47 to establish that square MHJN + square ACDB = square EHKF; thus rectangle ALOF = square ACDB and by subtraction square BNOF = rectangle LCDN.

What Euclid doesn't say and even more mysteriously what Heath doesn't say is that AB:AF is Ø, that ND:BD = Ø, that what we have found here is the extreme and mean ratio. Heath does point out the algebraic connection. If we set AB = a, and BF = x, then $a^2 - ax = x^2$, or $ax + x^2 = a^2$. We may take this further by setting a = 1 in which case we have $x + x^2 = 1$; x being equal to the

Euclid II.6 and II.11 - To divide a line into two unequal segments such that the square of the larger segment is equal in area to the rectangle whose sides are the whole line and the smaller segment.



1. Given line AB. Construct square ACDB.
2. Project AB to F, EF = ED. Construct square EHKF.
3. As per Euclid II.6, ALME is equal in area to NJKO. Thus rectangle ALOF plus square MHJN = square EHKF.
4. Since EF = DE then by Euclid I.47 squares ACDB + MHJN = square EHKF.
5. Therefore ACDB must = ALOF. Subtracting ALNB from both, square BNOF = rectangle LCDN.

Figure 3

Golden Section or Ø. If we set AB = to 1 then BN = Ø and ND = $Ø^2$. To satisfy the equations Ø or x must equal .618…

Euclid gets more to the point at VI.30, directly proposing 'to cut a given finite straight line in extreme and mean ratio'. He begins by restating II.11 as a given, 'let there be applied to square ACDB the rectangle ALOF equal to it in area and exceeding it in shape by the square BNOF'; and then proceeds directly to the proof. Just as one is tempted to say 'thanks a lot', a footnote, by Heath, not Euclid, refers us to VI.29. Here the problem is to construct on a given line a parallelogram that is both equal in area to a given rectilinear figure and 'exceeding' by a parallelogram similar to a given one. 'Exceeding' means that one side of the parallelogram is an extension of the given line.

Euclid begins VI.29 by dividing the given line at its center and on the half-line constructs a parallelogram similar to the given one. Then he constructs another larger similar parallelogram equal in area to the given rectilinear figure plus the parallelogram on the half-line. He has shown us how to do this at VI.25 where we construct a figure equal in area to one figure and similar in shape to any other; using techniques learned, in turn, at I.42, 44, and 45.

Then he places the larger parallelogram over the one on the half-line. This creates a gnomon which is equal in area to the original rectilinear figure. Since one leg of the gnomon is equal in area to the other [I.44] and one leg lies on the half-line, the parallelogram on the original line and its extension is equal in area to the original rectilinear figure. The corner of the gnomon is the 'exceeding' parallelogram.

Now, returning to VI.30 we apply the same method. But since our figure C is the square on given line AB and the 'exceeding' figure is to be similar to the very same square on AB, our square ACDB, we see that VI.30 is a special case of VI.29. Starting the same way we divide AB in half at E and create a square on EB, our square MHJN. To continue the method of VI.29 we ask what larger square can we construct whose area equals ACDB + MHJN. The answer is that sides of those two squares are two sides of right triangle EDB thus by I.47 the square on ED, EHKF is the larger square we seek. Subtracting square MHJN from EHKF leaves gnomon EMNJKF which we have seen at II.11 is equal to rectangle ALOF. The 'exceeding' square is BNOF. In our figure, AB = BD. BD:BN as BN:ND

Euclid returns to the subject of extreme and mean ratio at the beginning of book XIII, propositions 1 thru 5.

Our goal being more modest, merely to construct a rectangle whose sides are in the ratio of 1: Ø, we cut directly to the chase, so to speak, by applying line ED to AB, having no need of all the squares. But notice here that at II.11, where Ø is defined as a construction, it is not mentioned by name. And at VI.30, and later in book XIII, where Ø is named its construction is not specified. Even the reference to VI.29 is not specific, but a general case. Such obscurity is unusual in a work of towering lucidity like Euclid's.

It is significant that Euclid's proofs and methods all involve transformation of areas and that Ø is a special case of the transformation of the square at which both form and area are equal. There is an equality of transformation which we may symbolically relate to the experience of unity in multiplicity which we have elsewhere seen is a definition of beauty, the goal of any classical art. Looking deeper, beauty is the manifestation of Love, the force which binds the elements of the cosmos in harmonic order.

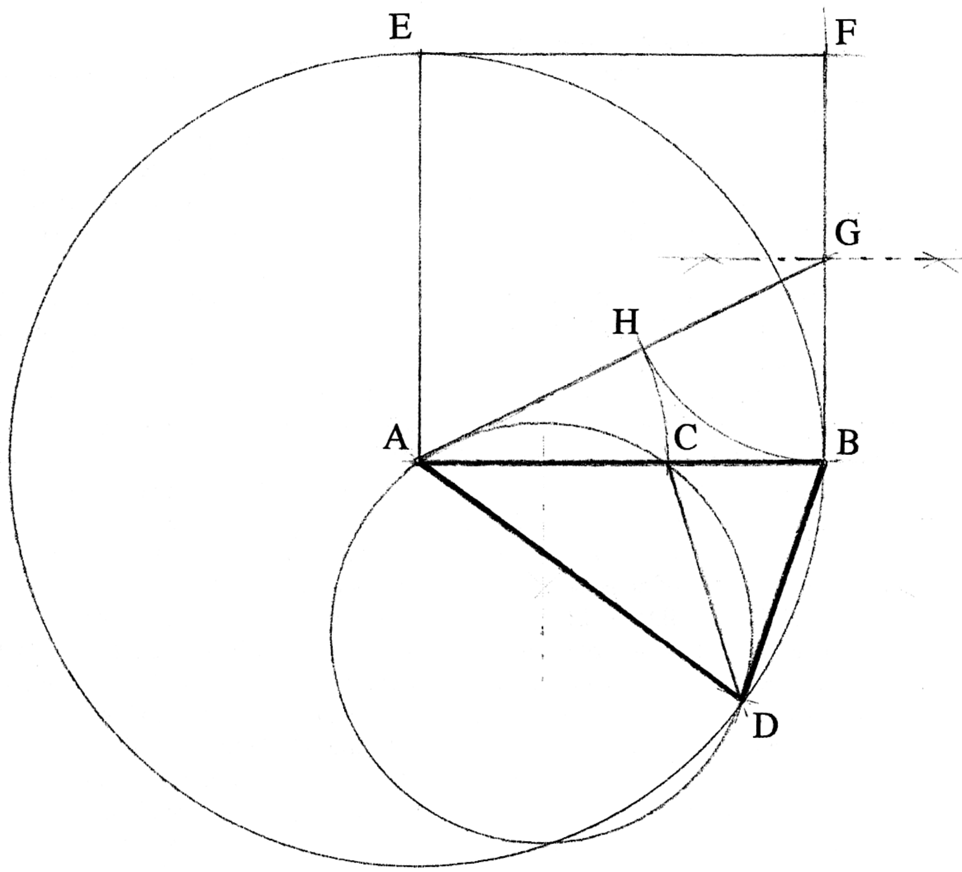
2.4 Construct a pentagon within a given circle 13

Euclid IV.10 - 11; see **figures 4 and 5**. Greenberg, p 22. Serlio p29. French 5.14, p 66. Lawlor, figure 5.4a, p 51.

Euclid now uses II.11 to construct an isocilies triangle whose apex angle is one-half of either of its base angles. The construction of this triangle is given at IV.10. The key is to divide a long side of the triangle, the given line AB, into Ø and $Ø^2$, at point C. AC = Ø, which will be the base of the triangle, line BD. Circle BDE is drawn with center A and radius AB. Triangle ABD is constructed within it, AD = AB, and BD = AC.

Euclid proves this is the triangle we seek by inscribing triangle ACD in circle ACD. He has told us how to do this at IV.5, similar to our construction 4.1. Then he shows that line DB is tangent to circle ACD by citing III.36 where we learn that if a point outside of a circle projects two lines, AB which cuts the circle and BD tangent to the circle, then the rectangle AB, BC is equal in area to the square on AC. Since this is the condition we have here at IV.10, as this equality of area is part of the definition of Ø, then BD is tangent to circle ABC.

Euclid IV.10 Construct an isosceles triangle having each of the base angles double the apex angle.

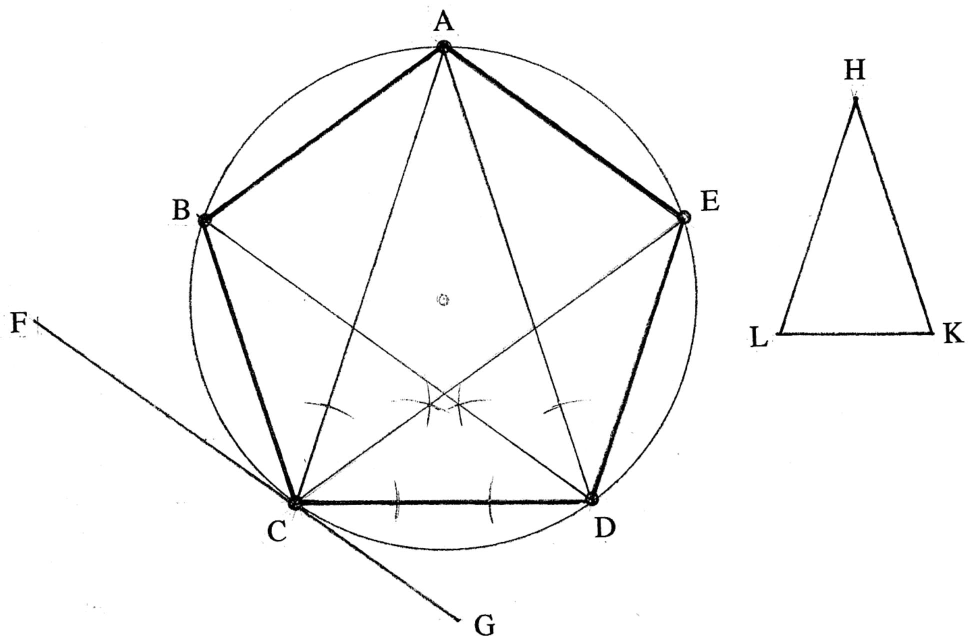


1. Given line AB, find the Ø point using the method of II.11. Construct square AEFB. Bisect FB at G. Draw line AG.
2. Lay off GB on AG to find H. Lay off AH on AB to find C.
3. Draw circle ABDE with center at A and radius AB.
4. With center at B and radius AC find point D on the circle.
5. Draw BD and DA to complete triangle ABD.

Figure 4

By III.32 angle BDC = DAC. By I.32, exterior angle BCD = the two opposite angles, CDA + DAC. Adding CDA to each we find that BDA = CDA + DAC = BCD, thus BDA = BCD. Since AD = AB, BDA = CBD =BCD. By I.6, sides opposite equal angles are equal, therefore BD = DC = CA and by I.5, base angles of an isosceles triangle are equal, CDA = DAC. Thus CDA + DAC = 2 x DAC = BCD, and BCD = BDA and DBA. Thus BDA and DBA each = 2 x DAC.

Euclid IV.11 Construct a regular pentagon in a given circle.



1. Given circle ABCDE.
2. Layout an isosceles triangle, HKL, with the base angles equal to twice the apex angle using the method of IV.10.
3. Draw line FG tangent to circle A-E at C.
4. Construct angles CDG equal to angle H and ACF = K.
5. Draw triangle ACD similar to HKL.
6. Bisect angles ACD and ADC. Project the bisectors to find points B and E. Draw lines AB, BC and AE, ED. ABCDE is a regular pentagon.

Figure 5

Returning to IV.11 we inscribe a triangle similar to HKL, which is similar to ABD in IV.10, in the given circle A-E. We do this by IV.2 which places line FG tangent to the circle at point C and uses this as a baseline on which to transfer the angles. Angle DCG = angle H, ACF = L, and thus angle ACD = angle K = 180^0. DCG = CAD by III.32. and ADC must = angle L.

To complete the pentagon we bisect angles ACD and ADC and project the bisectors to find points B and E. The proof is that using the properties of triangle ACD, angle DAC = ACE = ECD = CDB = BDA. By III.26, equal angles cut equal circle segments, thus the sides are equal. To establish that the angles are equal Euclid notes that AB = ED. Adding segment BCD to each, ABCD = EDCB. By III.27 angle BAE = AED and by extension all the interior angles are equal.

That the diagonals of the pentagon intersect at the Ø point is given by Euclid at XIII.8. This provides the possibility of setting up a Ø scale.

The construction of the pentagon we use at 2.4 is the standard draftsman's method. It is not given by Euclid but a version of it is given by Heath in his commentary on IV.11, attributed to a Mr. H. M. Taylor, our **figure 6**. Note that the construction gives both the pentagon and the decagon.

102 BOOK IV [IV. 11, 12

Mr H. M. Taylor gives "a complete geometrical construction for inscribing a regular decagon or pentagon in a given circle," as follows.

"Find *O* the centre.

Draw two diameters, *AOC*, *BOD* at right angles to one another.

Bisect *OD* in *E*.

Draw *AE* and cut off *EF* equal to *OE*.

Place round the circle ten chords equal to *AF*.

These chords will be the sides of a regular decagon. Draw the chords joining five alternate vertices of the decagon; they will be the sides of a regular pentagon."

The construction is of course only a combination of those in II. 11 and IV. 1; and the proof would have to follow that in IV. 10.

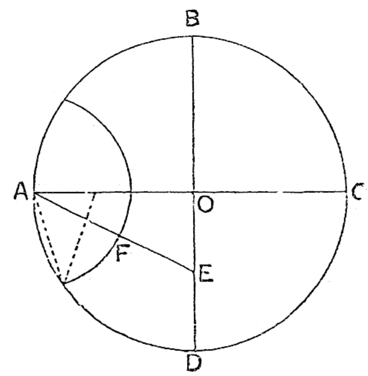


Figure 6

Heath concludes, "The construction is of course only a combination of those in II.11 and IV.1; and the proof would have to follow that in IV.10."

Thus Ø of the radius of the circle is the side of the decagon. If we set radius AO in Taylor's figure equal to 1, then AF = Ø. The method we use at 2.4 uses Ø of the radius applied to the diameter. If the diameter in Taylor's construction, AC = 1 then AF = $Ø^2$.

Euclid expands this idea of multiple polygon relationships at IV.16

A fascinating, occult relationship between the hexagon and the decagon within a given circle is given at Euclid XIII.9 and 10, by pointing out that the radius is equal to the side of the hexagon. Book XIII is devoted to the construction of the five Platonic solids and proposition 9 is necessary for the construction of the dodecahedron at XIII.17.

2.5 Construct a pentagon given a side 14

Not given by Euclid or Greenberg.

This construction is from the sixteenth century artist Albrecht Durer. Anecdote has it that he was expelled from the guild of engravers in Lubeck because of its publication. This would imply that matters concerning Ø were still held secret by various private organizations right into the beginning of the modern age.

The foundation of this construction is the Vesica Pisces and the Golden Section.

Session 3

3.1 Construct a hexagon given a side 15

Not in Euclid. Greenberg, p 22

Euclid does not give this construction but it is essentially the reverse of IV.15. Starting with equilateral triangle ABC we could simply construct five more of them; but here we pursue a more general case by finding point C which is the center of the hexagon's circumscribing circle. Finding the center of this circle is the key to constructing any regular polygon, beginning with a side.

The hexagon exhibits the remarkable property that the radius of its circumscribing circle makes exactly six intervals about its circumference. Consider the qualities of this figure if the circle marked by six points rotates about its center. The center point, being dimensionless does not undergo movement while the points on the circumference undergo rotation. This recalls the Biblical creation story of the six active days of creation and the seventh of rest. The extreme simplicity and directness of this figure's construction reinforce the symbolic association.

3.2 Construct a hexagon within a given circle 16

Greenberg, p 22. Serlio, p 28. Euclid IV.15. Euclid begins by drawing diameter BG, then uses the Vesica Pisces, figure ABCD, to establish the two equilateral triangles ABC and ADF. He then projects radii CA to F and AD to E. Since all the sides are equal to the radius of the circle the figure is equilateral. Since it is composed of equilateral triangles all of the interior angles are equal thus the figure is equiangular.

3.3 Construct an octagon given a side 17

Not in Euclid. Greenberg, p 22.

This construction is not given by Euclid although it is implicit in the idea of multiple polygons, being the double of the square.

3.4 Construct an octagon within a given square 18

Not in Euclid. Serlio, p 28. Benjamin pl III, fig 1. French, 5.15, p 66 Greenberg, p 22.

This construction is based on the half-diagonal of the square which equals the √2/2. Applied to the side of the square it creates a 3-4-3 cut that may be used in architectural composition; **figure 7**.

3.5 Construct an approximate heptagon given a side 19

Not in Euclid. Serlio, p 29. Benjamin pl. II, fig. 15. French, 5.16, p 66 Greenberg, p 25, alternate p 23.

Alternate methods of constructing polygons, and a discussion of the root rectangles are offered in **figures 8, 9, and 10**.

Figure 8 - The Vesica Piscis and the Origin of Form

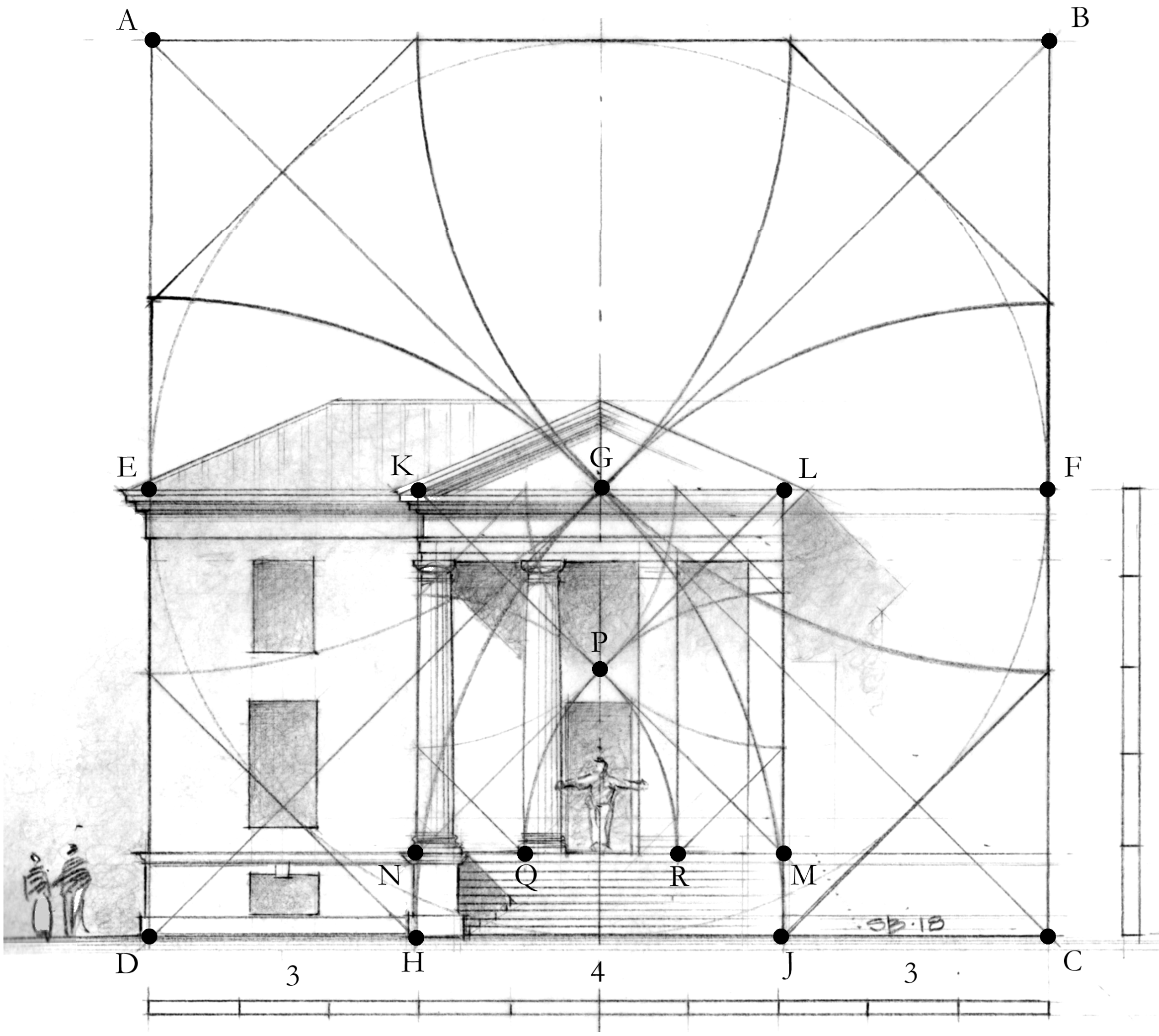
An ontogeny of the trans-rational ratios

Presented here is a geometrical ontogeny, a story of creation using number and geometry, one of an indefinate number of such stories, which gives rise to a series of 'transcendental' or 'trans-rational' ratios. That is, ratios that are not composed of whole numbers but of components that are not determinable in rational units. The most commonplace of such ratios is that of the circle's diameter to its circumference, known as pi, $\pi = 1 : 3.14\ldots$ The three dots after the numbers indicate that the sequence of numbers does not have a knowable termination but continues indefinitely. Pi has been calculated, not just to millions of digits but to millions of pages of digits with no end in sight.

These trans-rationals have a special character in that we can see directly the relation of the factors when they are drawn graphically, but we cannot fully express these relationships in rational mathematical [whole number] units. We may take this stubborn 'fact of life' as an indicator that the rational aspect of the human being, while important in the expression and communication of knowledge, is not the whole story. Or to put it another way, there may never be a fully rational explanation of existence and the cosmos, or as the humorist puts it, to 'life, the universe and everything'. This may account for the persistence of the power of myth. Certainly, in the stimulation of remembrance the artist must, in some way, enter the realm of the trans-rational.

We begin on the level of the Monad, with the mysterious point of emergence and its inflation into the circle, **figure 8a**. We project the initial ray of creation or lightening flash, **figure 8b**. This stage of Division, from the Monad to the Dyad could be represented by placing two circles within the circle of Theos, each half

An architectural design using the 'root 2' cut



1. Given square ABCD, center at G, and half-square EFCD.

2. With center at C, radius CG, draw arc GH. Repeat with center at D to find J.

3. Draw square KLMN with side = HJ, and center at P.

4. With center at M, radius MP, draw arc PQ. Repeat from N to find R.

Note that the whole, the elevation, and the part, the portico, are constructed on the same geometrical pattern

Figure 7

the diameter of the larger circle, forming a figure eight or meniscus. But we need to remember that both these circles, symbolizing the Same and Different or Idea and Matter, are aspects of 'Theos', and they divide within Theos, thus preserving a single unified cosmos.

In this story we must envision, as in **figure 8c**, that in response to the ray of creation the circle of the Same begins as the circle of Theos itself and withdraws upwards, decreasing in diameter, while maintaining its tangency at point A; and simultaneously the circle of the Different, also beginning as the circle of Theos, withdraws downwards diminishing in diameter while maintaining its tangency at point B. The differentiation continues towards the center C, stopping at a place where the center of one circle lies on the circumference of the other. The centers of these two circles may be found, **figure 8d**, by placing the circle of Theos in a square, DEFG and drawing the diagonals and semi-diagonals. Where these intersect, at H, J, K and L, the side of DEFG is divided into thirds. Points M and N give the centers of the circles of the Same and the Different.

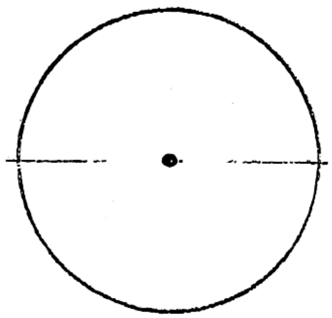
The football shaped area of overlap of the circles of the Same and Different, **figure 8e**, represents the area of Psyche, Soul or Mind; the Triadic level of our ontogenic story. This shape is called the Vesica Pisces or Vessel of the Fish. If we connect MN and PQ the lines will meet at C, the center of the circle of Theos, and will cross at a right angle, 90 degrees, **figure 8f**, symbolizing the Tetradic or fourth ontological stage. Because intervals MN, MP and NP are all radii, shape MNP is an equilateral triangle. If the distance MN, from center to center is taken as 1, the distance from point to point, PQ is equal to the $\sqrt{3} = 1.732\ldots$. The Vesica Pisces played a role in the life of early Christianity. As a graphic sign it is said to have marked Christian meeting places during the time when the religion was illegal. In a deeper way it is related to Christianity symbolically as the birth of Jesus occurred at the start of the Piscean age, when the sun on the Spring equinox began to rise in the constellation Pisces. For the previous two thousand years it had been moving through the constellation Aries; it is now about to enter the constellation Aquarius. The Christian era has thus taken place in the Piscean, or water age. Water imagery surrounds Jesus; he is the 'fisher of souls', his disciples are fishermen, he walks on water, and feeds the multitude with loaves and fishes. In John 21:11 the resurrected Christ bids fishermen to cast their net into the water and causes the miraculous catch of 153 fish. This is taken by some to be a reference to the Pythagorean 'measure of the fish' or 265/153, a whole number approximation of the $\sqrt{3}$, integral to the Vessica Pisces.

The Vesica Pisces may also be likened to the yoni or female birth organ. Just as we saw the Vesica Pisces give birth to the initial form, the equilateral triangle, it may also give birth to any regular polygon. The 'trick' being how far to separate the two radii, MP and NP. Stated another way, the task is to find the center of the circle which will circumscribe the desired polygon. For our purposes here, **figure 8g**, we construct a semicircle centered at C with radius CM and find point R. Extend a line through MR until it intersects the circle at S. We may repeat the process from N through R to point T. MNST is a square, and we note that the diagonal $MS = NT = \sqrt{2} = 1.414\ldots$.

Now it may be noticed that we have constructed our triangle and square on the right side of the Vesica Pisces, and that there is a symmetrical left side. If we construct a mirror image square on that side, **figure 8h**, we have rectangle UTSV. If ST = 1, VS = 2; and 2 we may note is the $\sqrt{4}$. By the Pythagorean theorem, $a^2 + b^2 = c^2$, we determine that the diagonal of double square UTSV is equal to the $\sqrt{5} = 2.236\ldots$. This represents a movement beyond the Tetradic story of descent. The Pentad, or fiveness, symbolizes the balance point, or we might see it as the 'turning point'. In the terms of our creation story we may say that the first four stages represent our descent from Unity into Materiality.

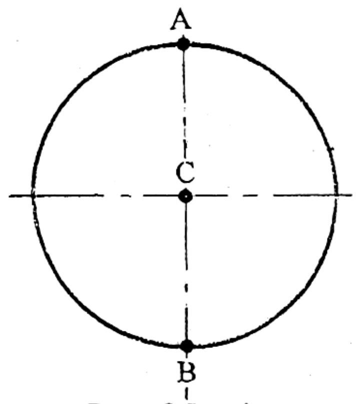
In the Pythagorean myth, utilizing the earth centered cosmos, the individual soul begins its journey by projecting itself outward from the divine realm through the sphere or shell of the fixed stars as a ray of light. As the psyche approaches the earth it passes through the shells of the seven planets, acquiring character. It passes below the level of the moon and begins to take on material form. At last it reaches the earth where physical birth occurs. But according to Plato the effect of the journey into incarnation is to cause the psyche to, in effect, become disoriented and to identify with the body and forget its identification with divine unity. Thus our earthly work is one of remembrance or in Platonic terms 'anamnesis'. Such remembrance being necessary for our return journey to the divine. This return from multiplicity to unity is symbolized in the ratio called phi, Ø, or the Golden Section. In **figure 8j**, we take the interval UC and 'apply' it along line AMC to find point W. $UC = \sqrt{5}/2$, and $MC = 1/2$, thus the numerical value of phi = .618… . If we draw a line through W parallel to UM, and extend UV the two lines will meet at point Y creating a phi rectangle.

Point Y falls on the circle of Theos, the Divine. This is because WC = 1.118… and YW = 1; $YC = 1.118\ldots^2 + 1 = \sqrt{2.249\ldots} = 1.499\ldots$ and this equals AC. Thus Ø has geometrically reenacted the metaphor of Platonic remembrance and returned us to Unity via a journey through multiplicity.



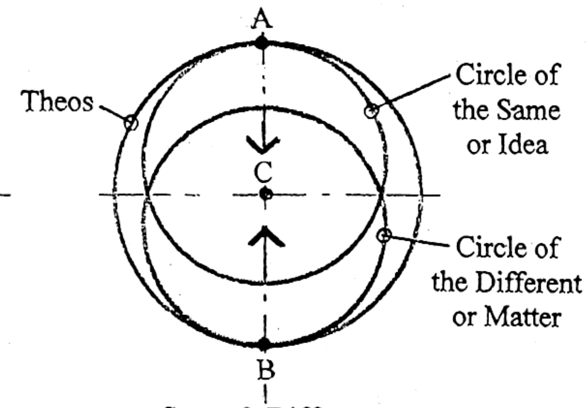
Circle of Theos

Figure a



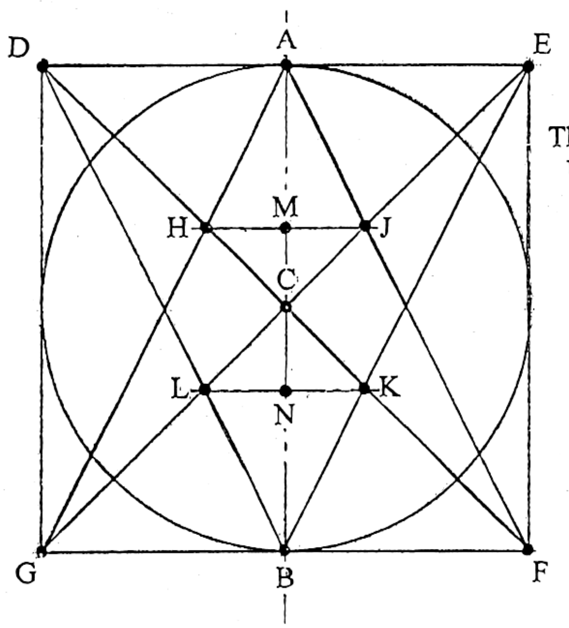
Ray of Creation

Figure b



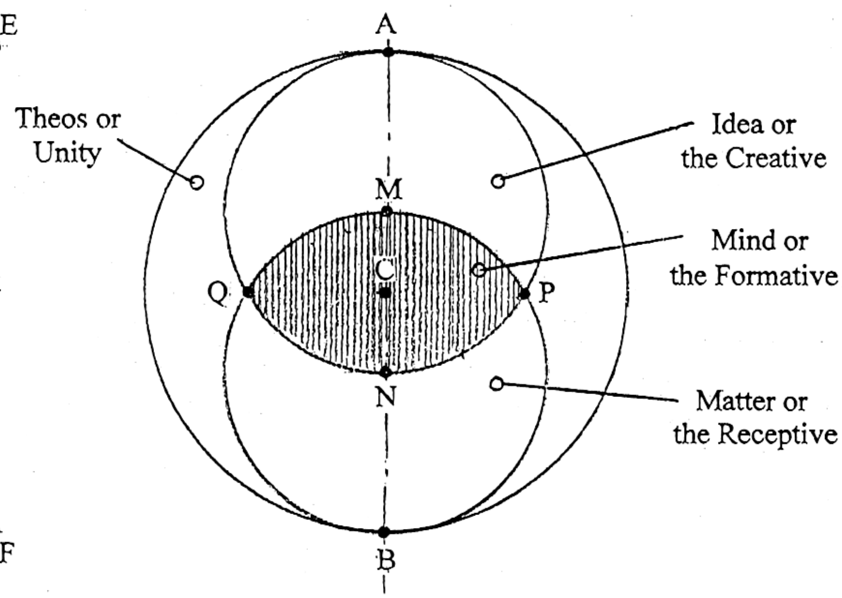
Same & Different Separate Within Theos

Figure c



Finding the Centers of the Vesica Pisces

Figure d

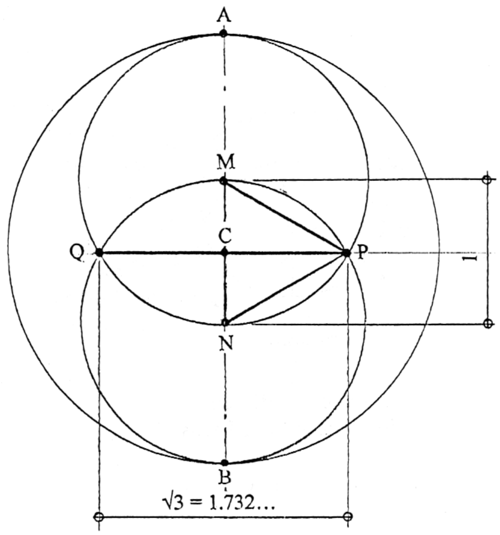


The Vesica Pisces

Figure e

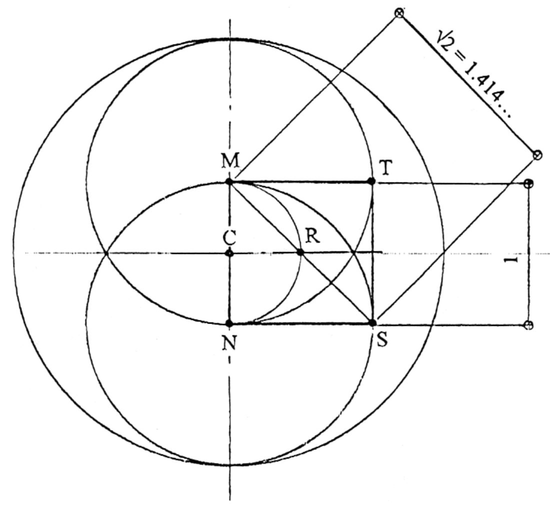
A Cosmology of the Vesica Pisces

Figure 8 a - e



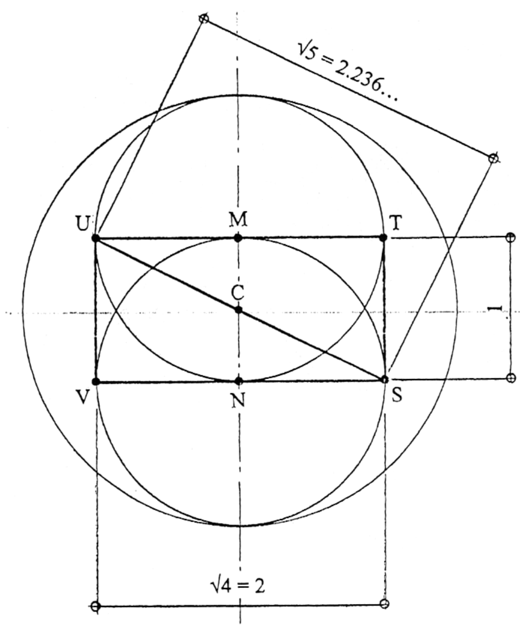
The Equilateral Triangle
and the √3

figure f



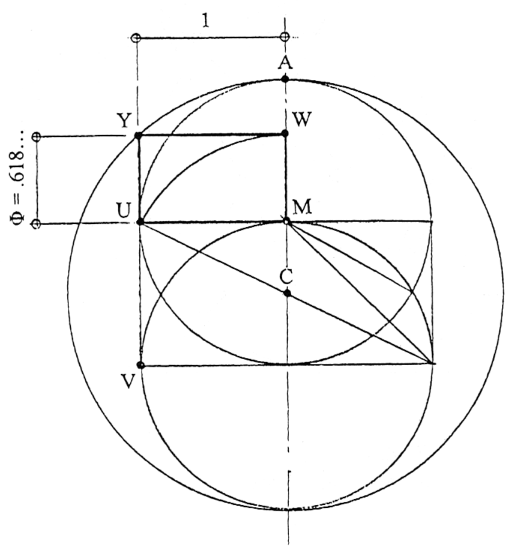
The Square
and the √2

Figure g



The Double Square
and the √5

figure h

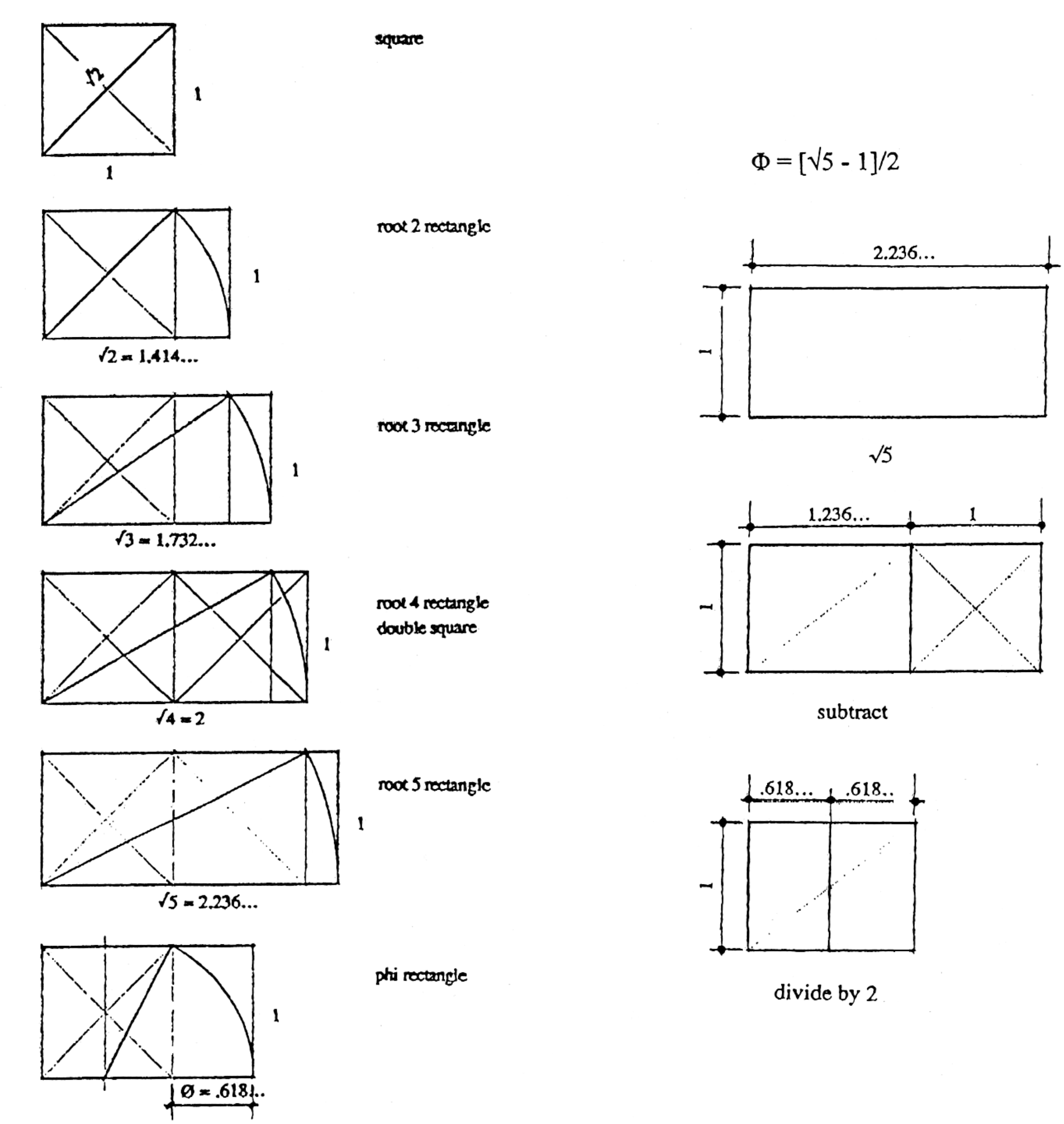


Phi Φ Returns to
Unity Through Multiplicity

Figure j

The Root Rectangles, Phi Φ, and the Return to Unity

Figure 8 f - j



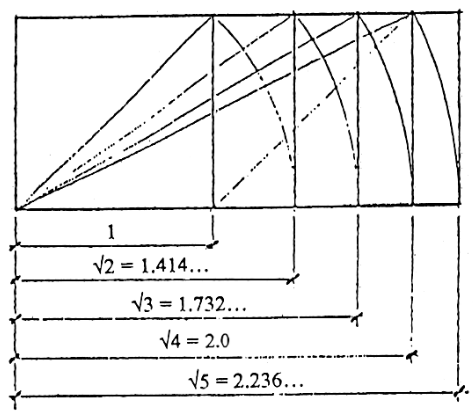
a. The Root Rectangles - Individual Construction

b. Geometrical Algebra of Ø

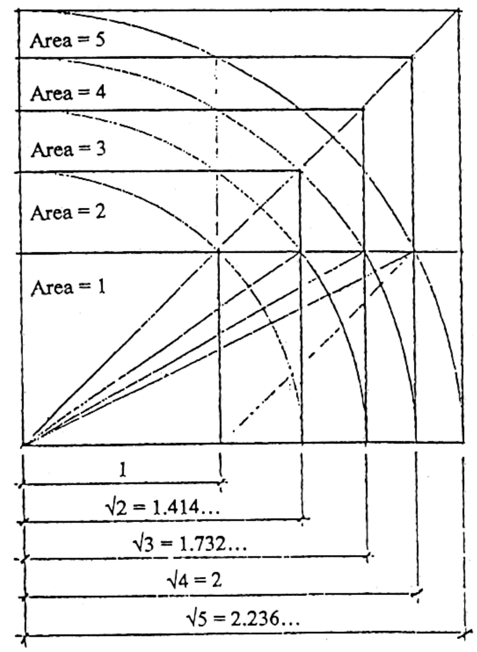
Figure 9 a - b

Figure 9 - Properties of the Root Rectangles

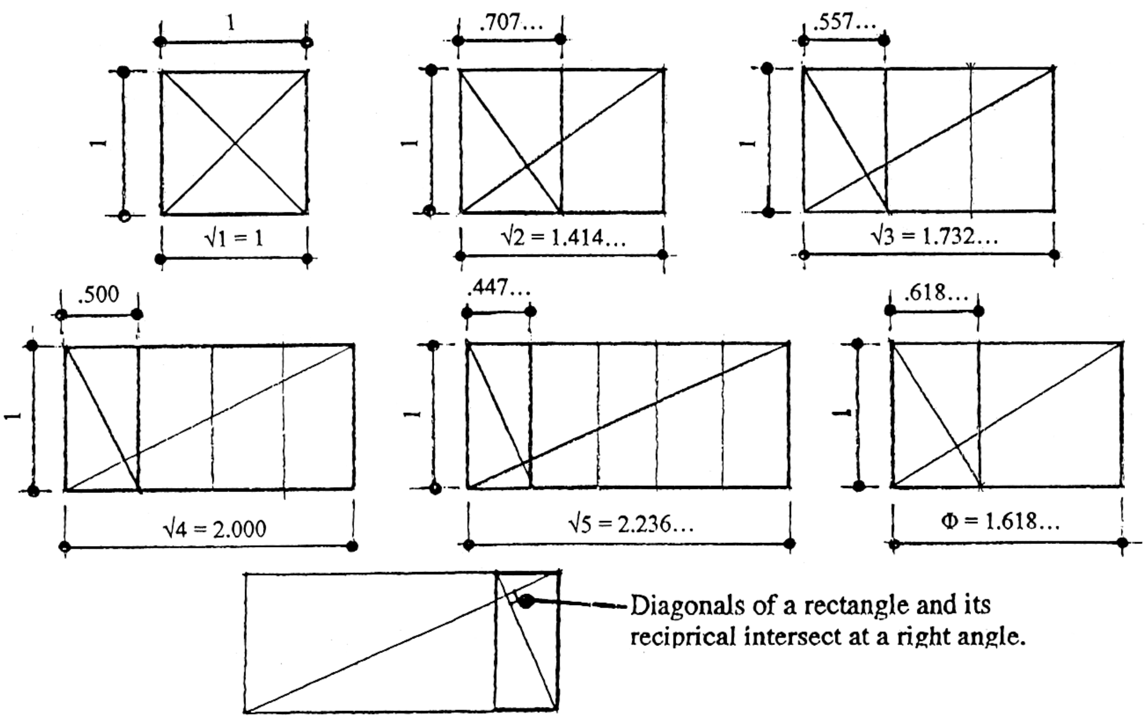
The root powers form a series of rectangles exhibiting a family-like resemblance, starting with the square, **figure 9a**. The diagonal of a square whose side =1 is √2, =1.414… Rotating the diagonal down to the horizontal, or as geometers might say, 'applying' the diagonal upon the extended edge, produces a rectangle whose sides are in the ratio of 1:1.414…, called a '√2' rectangle.



c. The Root Rectangles - Superimposed Construction



d. Area Relations of the Root Rectangles



e. Reciprocals of the Root Rectangles

Figure 9 c - e

In a remarkable aspect of the nature of number the diagonal of a √2 rectangle =√3, 1.732… Continuing with the process it is even more remarkable that the diagonal of a √3 rectangle is equal to √4=2, producing the double square. This constitutes a kind of 'crossover' point between the rational and the trans-rational ratios. Continuing another step, the diagonal of a double square =√5=2.236…

Although we could continue indefinitely the design tradition has not found it necessary go beyond the √5 rectangle, most likely because this is a source of the Golden Section, phi or Ø, as Ø=[√5-1]/2. A geometrical version of this algebraic procedure, **figure 9b**, starts with a √5 rectangle. Subtract a square, then divide the remaining interval by 2, leaving a Ø rectangle.

In a superimposed view of the root rectangle family, **figure 9c**, we see the 'crossover points' between the rational and trans-rational ratios marked by the square and the double square. An extension of this diagram, **figure 9d**, illustrates the area relationships of the family. The square, initial representative of unity, has an area of 1; a square whose side is equal to the $\sqrt{2}$ has an area of 2; and so on.

All rectangles have a 'reciprocal'; **figure 9e**, that is, if the short side of a rectangle is 1 and the long side is 2, if we then set the long side equal to 1 the short side is now equal to 1/2; or if the sides are 1 and $\sqrt{2}$ the reciprocal is 1 by $1/\sqrt{2}$, or 1:.707… The reciprocal of the $\sqrt{3}$ rectangle is $1:1/\sqrt{3}=1:$ .557…; of the $\sqrt{5}=1:1/\sqrt{5}=.447$… The reciprocal of a $\sqrt{2}$ rectangle divides its parent rectangle into two equal parts; the reciprocal of a $\sqrt{3}$ rectangle is 1/3rd of its parent; and so on. The reciprocal of a Ø rectangle is another Ø rectangle plus a square because 1 : .618… = 1.618… : 1. The square is its own reciprocal. The diagonal of a rectangle intersects the diagonal of its reciprocal at a right angle. This is called by Le Corbusier the 'place of the right angle'.rectangle intersects the diagonal of its reciprocal at a right angle. This is called by Le Corbusier the 'place of the right angle'

Figure 10 - The Regular Polygons
The Vesica Piscis

The circle, as we have seen, constitutes its own cycle of creation. It is a Monad, 'a world in itself'. This Monad contains within itself its own opposite, and so divides itself from within itself to create the Dyad. Geometrically, this may be represented by giving rise to another circle, shown in **figure 8a-c**. But this new circle can only be experienced if it moves away from its twin.

Following Critchlow's description [IP, p14]: *"As that which is departing or 'externalizing' from the original circle proceeds on its course, a series of significant arcs are formed. The most significant of these in our present context is the half-way point, the last position at which the departing circle has contact with the centre of its origin. This position holds essential symbolic value inasmuch as it represents a union of an origin and a manifestation where the centres of both coincide with the peripheries of each, and the amount manifest of the departing circle exactly reflects the amount remaining of the original one. Also the departing circle has moved away by exactly the same amount as the original circle had 'moved away' from its hidden centre point in order to establish itself."*

Lawlor is even more direct [SG, p32], *"As the initial circle (Unity) projects itself outward in a perfect reflection of itself there is an area of overlap defined by the two centres and the intersection of the two circumferences. This area and shape is known as the Vesica Piscis."*

The Latin term means 'bladder of the fish', or in Lawlor's words [SG, p31], *"a bladder (Vesica) which when filled with air would be in the form of a fish (Piscis)...."*

In **figure 8e** we see the two circles and their relationship. The upper one can be considered the realm of the heavenly, or Ideal. The new lower one, the realm of the material, a reflection of the heavenly. Their overlap, the Vesica Piscis, is the realm of the created, or of formation. It is the psyche, the imaginative, mental world in which the material is shaped by the ideal.

As Lawlor puts it [SG, p32], *"One of the ways to view the Vesica Piscis is as a representation of the intermediate realm which partakes of both the unchanging and the changing principles, the eternal and the ephemeral. Human consciousness thus functions as the mediator, balancing the two complementary poles of consciousness."* Going further, he adds [SG, p33], *"The overlapping circles - an excellent representation of a cell , or any unity in the midst of becoming dual - form a fish shaped central area which is one source of the symbolic reference to Christ as a fish. Christ, as a universal function, is symbolically this region which joins together heaven and earth, above and below, creator and creation."*

The Vesica and Multiplicity

The Vesica, or psyche, in its 'formative' role is the birth giver to multiplicity. It can also be thought of as the 'yoni' of the Vedic tradition, a symbolic vagina, through which the living, material world is born. In the constructions that follow, the primary regular polygons of three, four and five sides are generated directly from, or 'born out of', the Vesica. Other polygons are generated from the Vesica and some combination of the properties of the primary polygons. The constructions used are designed to serve in the illustration of this metaphor and are not the only, or even the most efficient, methods of obtaining the polygons; as the saying goes 'Paths are many, truth is one'. The qualitative properties of the numbers involved are discussed in Taylor's 'Theoretic Arithmetic', and Waterfield's 'Theology of Arithmetic'.

Oneness, the Point-Circle. This is the circle of Theos or Unity in **figure 8a-b**. For brevity's sake it will not be shown in figure 10 but the reader should remember it as the origin and containment of form.

Twoness, the Line. The Vesica is defined by the two centers of its parent circles and by the two points of intersection of their boundaries. If the centers are connected by a line, this may be taken to represent a vertical axis, **figure 8e**. By connecting the points of boundary intersection a horizontal axis is generated, **figure 8f**. We may also see here that the cardinal points of earth measure, North, South, East, and West, arise directly from the Vesica. The two axes meet at a new center. Where they cross they create the 'right' angle. It is 'right' in the sense of being 'upright' or balanced between the poles of the Dyad. It is like the arithmetic mean, equal in magnitude between the two extremes.

Primary Polygons. Three, five and seven are prime numbers. They can only be acheived by addition of ones, not by multiplying any other numbers. Although four is not a prime number it is considered 'primary' in its form as the square. While seven is a prime number it may be thought of as a composite polygon.

Threeness, the Triangle. The Dyad is in constant fluctuation. Stability or structure requires a Triad, so the minimum number of edges a regular polygon or any polygon can have is three. The triangle may be considered symbolic of fixed structure because it cannot change shape without also changing the length of one of its sides. No other polygon has this property. The triangle created in **figure 8f and 10a**, is known as an equilateral triangle because all three of its sides are the same length, and each of its vertices are equal in angle at 60 degrees. There are two other types of triangles, isosceles, with two equal sides and scalene with unequal sides.

It is fair to consider the equilateral as first among them because of its closer adherence to Unity. In the Theology Of Arithmetic [p114, trans by Waterfield] we read, *"The first triangle is the equilateral, which has in a sense a single line and angle - I say it is single, because its sides and angles are equal, and what is equal is always indivisible and uniform. The second triangle is the half square, which has a single distinction of lines and angles, and so is seen in terms of the dyad. The third is the half triangle (i.e. half an equilateral triangle): it is altogether unequal in each respect, so from all points of view its number is three."*

To construct the first or primary polygon, the triangle, shown in **figure 10a**, we start with interval AB [MN in **figure 8f**]. This line will be the standard unit side or baseline of each polygon in this series. The apex of the triangle, point C, is where the two circle boundaries meet. Connecting the two centers and the boundary intersection completes the construction.

Fourness, the Square. Next is the four sided figure, the square or tetragon, shown in **figure 8g and 10b** . This is created by starting at points A and B [M and N in **figure 8g**], and rotating the two sides of triangle ABC in opposite directions along the circumferences of the two circles of the Vesica. The rotation stops when the two sides are at 'right' angles to the base, at points F and G. To find these draw arc with center at D, radius DA to find L. line AL and extend to find G; and DL to find F. Draw square ABFG. The circumscribing circle with ceneter at L, radius AL, may be used to find F and G but is not necessary.

It may be said that as the square is composed of four equal lines and four equal angles, it is a single fourness of four. The right angle is one fourth of a complete rotation; 360/90 = 4. There are four of them in the square, adding back up to a complete rotation. The square, then, is a multiple symbol of rightness.

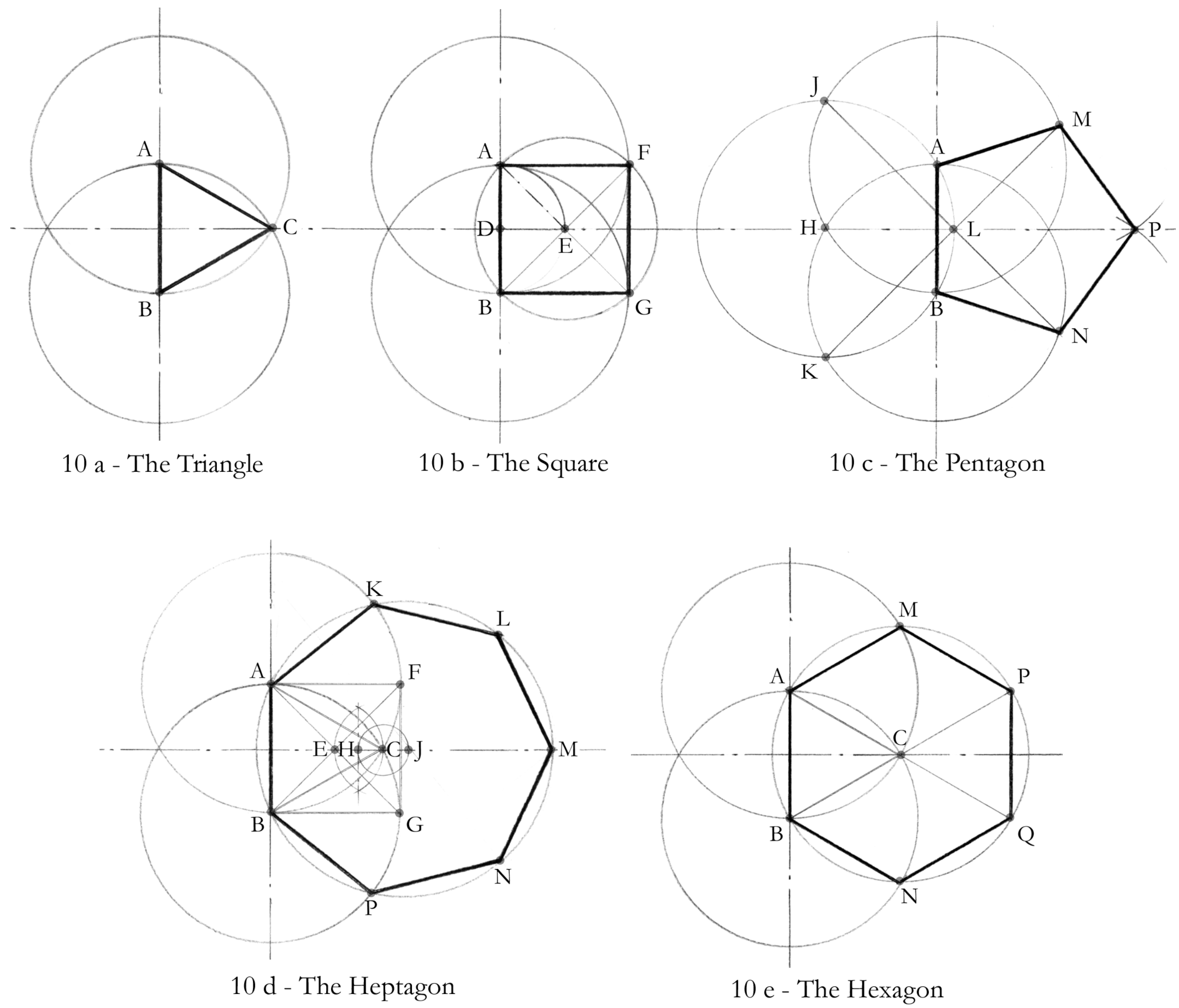


Figure 10 a - e

People who strictly follow social norms are sometimes referred to as 'squares', and those who adhere to virtue are known as 'upright'.

The square, like its inner aspect, the cross, is also associated with the realm of the material. Lawlor says of this [SG, p24], *"By definition, the square is four equal straight lines joined at right angles. But a more important definition of the square is the fact that any number, when multiplied by itself, is a square. Multiplication is symbolized by a cross. When we cross a vertical with a horizontal, giving these line movements equal units of length ... we see that this crossing generates a square surface: a tangible, Multiplication is symbolized by a cross. When we cross a vertical with a horizontal, giving these line movements equal units of length ... we see that this crossing generates a square surface: a tangible, measurable entity comes into existence as a result of crossing. The principal can be transferred symbolically to the crossing of any contraries such as the crossing of male and female which gives birth to an individual being, or the crossing of warp and weft which gives birth to a cloth surface, or the crossing of darkness and light which gives birth to tangible, visible form, or the crossing of matter and spirit which gives birth to life itself. So the crossing is an action principle which the square perfectly represents. The word nature means, 'that which is born' and all birth into nature requires this crossing of opposites. So the square came to*

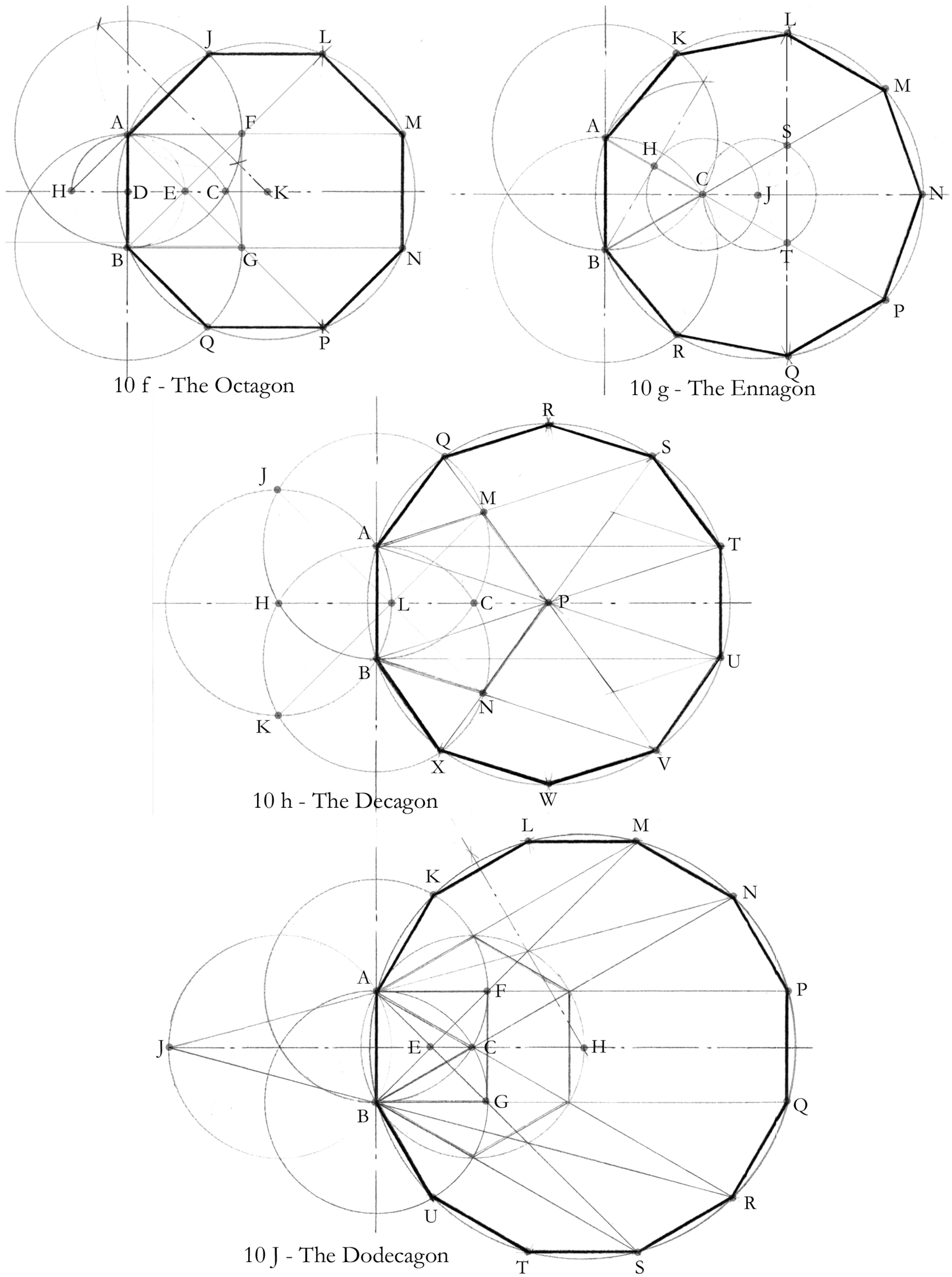


Figure 10 f - j

represent the earth and as such symbolized the conscious experience of finite existence, of that which is born into nature." Arithmetically the square is 1 x 1 = 1.

Fiveness - The Pentagon. The five sided regular polygon may be generated as shown in **figure 10c**. Construction of the pentagon starts with the same baseline, AB, used in the previous constructions. Again, the baseline will be rotated from both of its ends A and B, which are the centers of the Vesica circles, to form two more sides of the polygon. The question, as with the square, is where to stop the rotation of the radii.

This is found by constructing a circle with center at H, radius HB = AB to find points J and K. Project lines from the apexes of the two new vesicas, points J and K, through the point L, where the boundary of circle H crosses the central axis of the original vesica. Extend these lines until they cross the boundaries of the original vesica circles at points M and N. These are the points at which to stop the rotation of the sides. The two remaining sides of the pentagon are positioned by setting the compass to the length of the side AB. With centers at M and N, redius AB find point P, where the radii intersect the center axis of the Vesica. Connecting this point to the top of each side completes the pentagon.

This process of utilizing the Dyadic nature of the original Vesica plus the Triadic nature of the expanded triple Vesica yields the fivefold figure of the Pentagon, as in 3 + 2 = 5. This construction was first made public by Albrecht Durer in the fifteenth century, though it is probably much older.

Seveness The Heptagon. Construction of the seven sided polygon starts from the same base line AB, the line connecting the centers of the Vesica circles. This will provide the location of the first two vertices. In the remaining constructions we will find the circumscribing or 'controlling' circle, on the boundary of which all the vertices of the polygon will fall. The key to the constructions will be to locate the center of this circle.

The seven sided polygon, or heptagon, **figure 10d**, is constructed starting with the Vesica, the triangle, and the square, similar to **figure 10a & b**. To find the center of the controlling circle draw the diagonals of the square, to find point E, and the apex of the triangle, point C. Bisect EC to find H. With center at C, radius CH find point J, the center of the controling circle. Then, using the compass, lay off a distance equal to this, above the apex of the triangle E. This is the center point of the heptagon's controlling circle. Find vertices K thru P by setting compas at radius AB and 'walking the intervals on the circumference of the controling circle.The first two vertices are known from the baseline AB. The second two, K and P occur where the controlling circle crosses the Vesica circles. The fifth is where the Vesica axis crosses the controlling circle H. Note that this is an approximate construction. The Heptagon, then, is realized through utilizing the properties of the three and the four, similar to the arithmetic 4 + 3 = 7.

Composite Polygons. The triangle, square, and pentagon may be said to arise directly from the Vesica, as has been shown. The hexagon, octagon, decagon [10 sides] and the dodecagon [12 sides] may be thought of as 'composite' in the sense that they can be generated through combinations of the properties, the sides and diagonals of the 'primary' polygons.

The Hexagon is constructed from the doubling or reflection of the triangle, as in 3 x 2 = 6, **figure 10e**. It can also be seen as created by six movements of the triangle, each of which is 1/6 of a rotation, or 60 degrees x 6 = 360. The circumscribing circle is centered at C, raduis AC.

The Octagon is constructed using the diagonals of the square, **figure 10f**. Construct square AFGB. With center at D, radius DA find point H. Draw line AH and project it to find J. Bisect AJ and extend the bisector to find K, the center of the circumscribing circle. Extend the square's diagonals to find L and P. extend the sides of the square to find M and N. Here, the properties of the square and its diagonals are utilized to generate the eight sided figure, similar to the arithmetic 4 x 2 = 8.

The Ennagon or nine sided polygon, **figure 10g**, is constructed starting with the Vesica and the triangle ABC. Bisect AC to find point H. With center at C and radius CH find point J, the center of the controlling circle. The first two vertices of the ennagon are A and B. The second two fall where the controlling circle crosses the Vesica circles, points K and R. The fifth vertex is where the axis of the Vesica crosses the controlling circle, point N. The sixth and seventh fall where the projections of the sides of the triangle cross the controlling circle, points M and P. The last two can be found with the compass or, more interestingly, by drawing a circle centered at J, of radius JC = HC, half the side of the original triangle. Where the projections of the sides of the original triangle ABD, cross this circle at points S and T, draw a line, which will be parallel to the baseline AB, and extent it until it crosses the controlling circle at points L and Q. These are the last two vertices of the ennagon. This last line also creates two triangles LMS and PQT, similar to the original triangle ABD, who's bases coincide with every third side of the ennagon; as in the arithmetic 3 x 3. This is an approximate construction.

The Decagon or ten sided polygon, **figure 10h,** begins with the Vesica and pentagon, as in **figure 10 e**. The center of the controlling circle is the apex of the pentagon, point P. Points Q through X are found by projecting the sides and diagonals of the pentagon. Notice that a 'shadow' pentagon is formed around points P, T and U. The decagon, then, is created by a doubling of the pentagon, as in the arithmetic 5 x 2 = 10.

The Dodecagon. The twelve sided polygon, the dodecagon, **figure 10j**, begins with the Vesica, the triangle and the square. With center at C and radius CA find point H, the center of the circumscribing circle. The initial vertices are A and B. The remaining vertices may be found by various projections of sides or diagonals of the square and triangle. Here, the properties of the three and the four are utilized to generate the twelve, as in the arithmetic 3 x 4 = 12.

Prime Polygons

The eleven sided polygon and those with more than twelve sides are omitted here both for the sake of brevity, and due to their absence from the classical design tradition. An approximate construction of the eleven sided polygon is given by Critchlow [Islamic Patterns, p169]. Others of higher sidedness can be found in various contemporary textbooks or can be worked out from the principles of the Vesica and the primary polygons, the triangle, square, and pentagon.

We may see in this series of constructions the suggestion of a sort of geometric creation myth: beginning at the mysterious point of emergence; the expansion of this seed into the Monadic circle; the division of the Monad from within to form the Dyad; the Dyad's 'co-creation', so to speak, of the Triad in the form of the Vesica. From this triplicity the indefinite multitude of polygons is generated.

Additional Constructions of the Pentad and Heptad

Figure 11 - Construction of the Pentagon within a Vesics Piscis, or Given a Diagonal

Sevenness: The Character of the Heptad

by Steve Bass

January 2000 Revised August 2015, January 2019

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MUUP	Theon of Symrrna,	**Math Useful for Understanding Plato**, trans by R&D Lawlor, Wizard's Bookshelf, 1978

Arithmetic Character of the Heptad

Sevenness, the Heptad, as a prime number is not created by the multiplication of any other numbers, it is only created by the addition of ones. Nor do multiples of it create any other number within the decade; as 7 x 2 = 14. The other primes below ten do multiply within the Decade; 3 x 2 = 6, 3 x 3 = 9, and 5 x 2 = 10.

Theon [MUUP, p 68] gives a more complete picture, *"Among the numbers contained in the decade, some create and some are created, for example, 4 multiplied by 2 creates. Others are created but do not create, like 6 which is the product of 2 by 3 , but which does not create any other numbers in the decade. Others create but are not created, such as 3 and 5 which are not created by any conbination of numbers, but which create: 3 produces 9, and multiplied by 2 produces 6, and 5 multiplied by 2 produces 10."*

Construct a Pentagon within a Vesica Piscis, or given a diagonal.

1. Given interval AB and Vesica Piscis ABCD, with center at E.

2. With center at E and radius AE draw arc to find point F. With center at F and same radius find point G. EG = AB.

3. Draw line BG.

4. With center at G and radius GF find H.

5. With center at B and radius BH find points J and K. With center at K and radius KB find point L.

6. AJBKL is a pentagon.

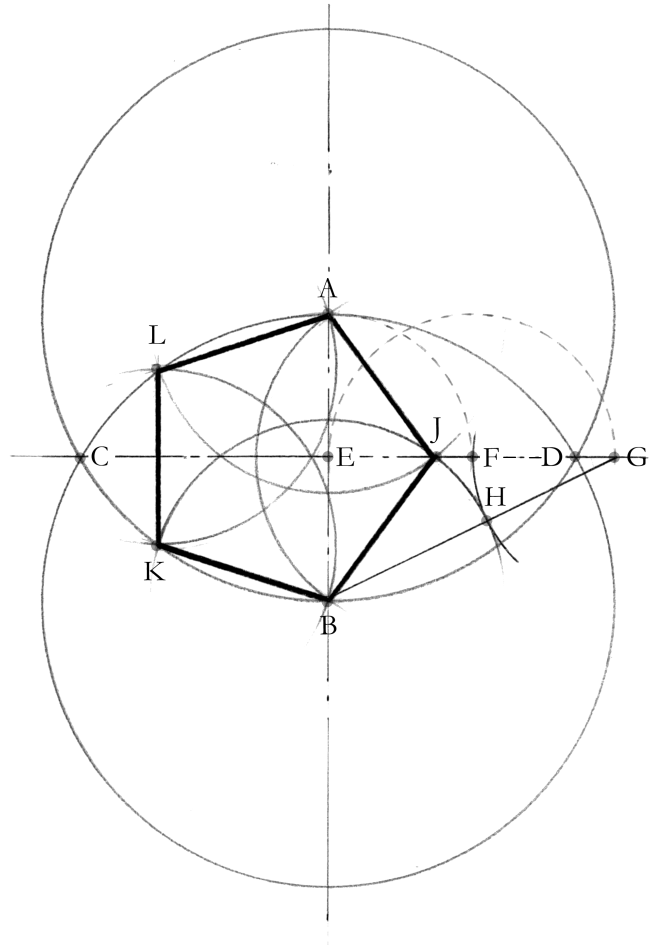


Figure 11

The Heptad may be considered to have a 'virginal' chatacter in the sense that it does not ingage in muptiplication within the decade. Theon [MUUP, p68] associated this virginal arithmetic quality of seven with the goddess Athena *"because this goddess was not born out of a mother and gave birth to none."*

As Graves recounts the myth [GM, p 46], *"... Athena's own priests tell the following story of her birth. Zeus lusted after Mitis the Titaness, who turned into many shapes to escape him until she was caught at last and got with child. An oracle of Mother Earth then declared that this would be a girl child and that, if Metis conceived again, she would bear a son who was fated to depose Zeus ... Therefore, having coaxed Metis to a couch with honeyed words, Zeus suddenly opened his mouth and swallowed her ... In due process of time, he was seized by a raging headache ... so that his skull seemed about to burst ... Up ran Hermes, who at once divined the cause of Zeus' discomfort. He persuaded Hephaestus, or some say Prometheus, to fetch his wedge and beetle and made a breech in Zeus' skull, from which Athene sprang, fully armed, with a mighty shout."*

A connection of this story to the arithmetic character of the Heptad is made in the TOA [p 99], *"They called the Heptad Athena ... because it is a virgin and unwed, just like Athena in myth, and is born neither of mother [i.e. of even number] nor of father [i.e. odd number], but from the head of the father of all [i.e. from the monad, the head of number]; and like Athena it is not womanish, but divisible number is female."* In Pythagirean terms even numbers have a female nature and odd numbers have a male nature.

The Heptad in the Four Subjects

In post-Platonic times higher education was divided into seven Liberal Arts. 'Liberal' in the sense that their study could help liberate psyche, the mind, from entrapment in matter. They were grouped into three verbal subjects, Grammar, Rhetoric and Dialectic; and four mathematical subjects Number, Geometry, Music and Astronomy. There were seven 'muses', or inspirational figures, each associated with a particular subject.

Arithmetic. We have seen above that seven, as a prime, cannot be created by multiplication within the Decade; nor does it multiply within the Decade. Thus it acquires its primary image as the Virgin number. Seven as 4 + 3 is related to the triadic cycle of proceeding, maintaining and returning [or we night see the triad as the three axes of measurenent]: plus the four stages of coming into being: point, line, plane and solid. As Taylor puts it [TA, p197], *"The forms or parts of the soul being three, viz, the intellective, the irasible and the epithymetic, four nost prefect virtues are produced belonging to these parts, just as of the three intervals [length, breadth and depth] there are four boundaries in corporeal increase. [viz. a point, a line, a superficies or surface, and a solid]"*

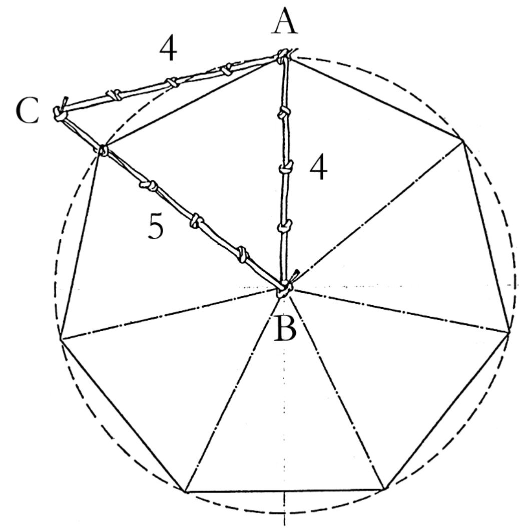


Figure 12a - Heptagon made with a knotted chord of 13 intervals.

It may also be noted here tht the sum of the numbers 1+2+4+4+5+6+7 = 28. 28 is a 'perfect' number in the Pythagorean sense because it is the sum of its factors: 1+2+4+7+14 = 28.

Geometry. Figure 12. Two methods of constructing the heptagon that could have been performed in the stone age are illustrated. The first, **figure 12a**, utilized a chord knotted into 13 equal intervals. We are more familiar with the chord of 12 intervals which can create the 3-4-5 right triangle. Here the more mysterious 13 is used to lay out the heptagon.

The second, **figure 12b**, is presented by non-orthodox Egyptologist Alexander Badaway [Ancient Egyptian Design Methods]. Given an isosceles triangle with base 8 and altitude 5. Bisect side AB and extend the bisector to find point D, the center of a heptagon's circumscribing circle. Extend the bisector to find point E. AE is a side of a heptagon. Note that as 4+4=8, the progression 5, 8, 13 is part of the Fibonacci series of numbers whose ratios approach the Golden Section. This method symbolically links seven to the 'higher' power of Ø as opposed to the other methods shown here which create seven from 'below', so to speak.

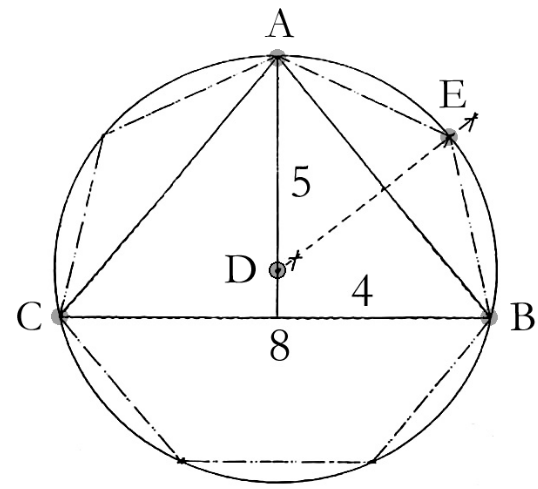


Figure 12b - The Heptad and Ø.

Figure 13. shows the construction of the seven sided polygon of Heptagon that begins with the Vesica Piscis 123c. Placing the compass open to the same radius at 1 draw arc a32b. Now having drawn three vesicas, draw their center lines, 1c, 2a, and 3b. Where they intersect at O draw a circle with radius O1. From point d on the circle's circumference draw arc d4 to intersect the circumference at e and f, de and df are sevenths of the circle's circumference. Note that we could have started with a given circle and

found points 1, 2, 3 and d by inscribing a hexad within the circle. Thus this construction symbolically created seven as a prime from the addition of six and one.

Figure 14 shows a construction of the Heptaon within a given circle. Draw diameter AOG and radius OF at a right angle to AOG. Draw the vescia AOBC. Draw lines AO and BF intersecting at D. From B draw an arc with radius BD to intersect circumference at E. Ar EF is one seventh of the circle's circumference. Note that AFG is a half-square and that BC is the side of an equilateral triangle, BCG. Thus this construstion symbolically creates seven from the interaction of three and four.

Figure 15 shows the construction of the Heptagon using a chord knotted into 13 intervals. The chord is shaped into an isosceles triangle with sides of 4, 4 and 5.

It should also be noted that all of the above constructions are approximate. Your present author does not believe that any exact constructions of the Heptagon are known.

Music. there are seven intervals between the eight notes of the musical octave. The musical fourth is positioned by the ratio of 4:3. The TOA [p 93] noted that 7 x 5 = 35 and that 35 is the sum of 6+8+9+12 which are the lowest whole numbers that describe the string lengths of the Pythagorean musical tetrachord; 6 is the fundamental, 8:6 is the ratio of the fourth note, 9:6 is the ratio of the fifth note and 12:6 is the ratio of the octave.

Astronomy. The Heptad is a key to the symbolisn of the astronomical heavens. There are seven 'planets' or wandering stars in the ancient cosmology. In order of their relative speed as seen from Earth they are: the Moon or Selene, Mercury or Hermes, Venus or Aphrodite, the Sun or Apollo, Mars or Ares, Jupiter or Zeus and Saturn or Chronos. In Pythagorean mythology these seven planet divinities form concentric shell domains around us through which the soul of the soon to be born person must descend from its origins in the divine realm beyond the fixed stars. The relative positions of the planets at the time of birth or the 'fall' were said to govern the formation of the personality.

The Moon in particular is associated with the number seven. The TOA [p 90] says there are seven phased of the moon: waxing crescent, first half, waxing gibbous, full, waning gibbous, last half and waning crescent. The new moon not being counted, it may be supposed because it is not visisibly there. The average length of the complete moon cycle is 28 days. This naturally breaks down into four parts of seven days, or weeks. The female menstrual cycle, and other human bio-rythyms, are keyed to the lunar cycle of 28 days.

The Heptad also provides a key link between music and astronomy. The seven intervals of the octave and the seven planets nay be combined to form a 'cosmic monochord' as shown tn the famous illustration, **figure 16**, from Robert Fludd published in 1617.

The Heptad in the 'Ages of Man'

The TOA [p 87] quotes Hippocrates as saying, *"Seven are the seasons, which we call ages - child, boy, adolescent, youth, man, elder, old man. One is a child up to the shedding of teeth, until 7 years; a boy up to puberty, until twice 7; an adolescent up to the growth of the beard, until three times 7; a youth during the general growth of the body, until four times 7; a man up to one short of fifty years, until seven times 7; and elder up to 56 years, until 8 times 7; from then on one is an old man."* In contemporary terms we might call these phases: birth to 7 - infancy, 7 to 14 - childhood, 14 to 21 - adolescence, 21 28 - youth, 28 to 35 - maturity, 35 to 42- late maturity, 42 to 56 middle age, and 56 to 70 old age. Traditionally the human lifespan is 70 years or 10 times 7.

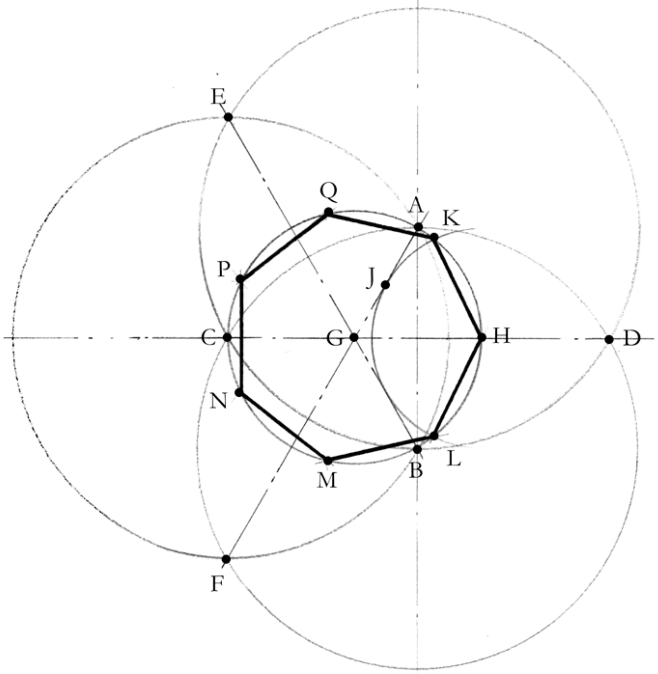


Figure 13

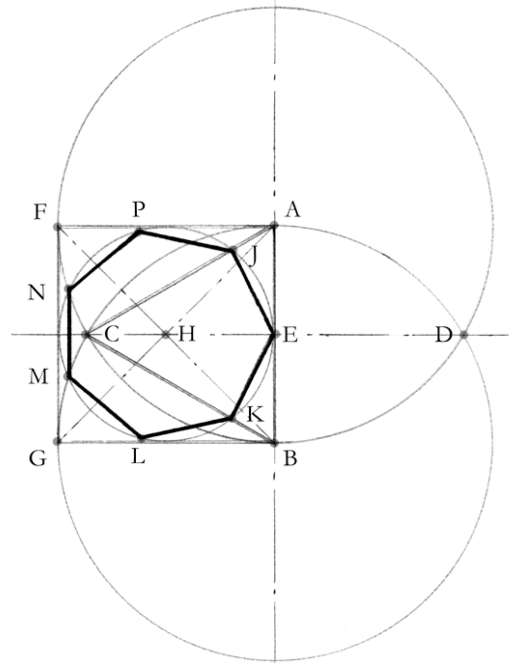


Figure 14

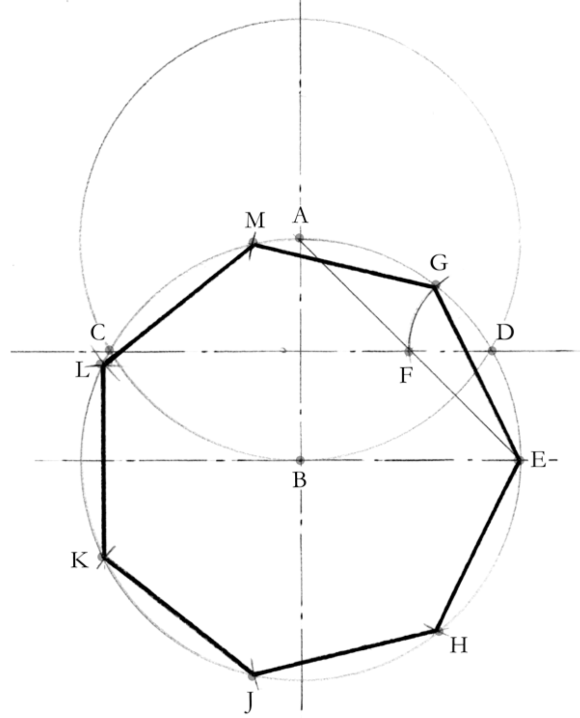


Figure 15

The TOA [p 94] adds, *"Children cut their teeth at seven months, at twice seven sit up and gain an unswaying posture, at three times seven they begin to articulate speech and make their first efforts at talking, at four times seven they stand without falling over and try to walk, and at five times seven they are naturally weaned and milk ceases to be their food."*

The TOA continues [p 95], *"In the third hebdomad [ages 21-28], they generally conclude growth in terms of length, and in the fourth they complete growth in terms of breath, and there is no other bodily increase remaining to them, for 28 is a complete number* [a Pythagorean perfect number]*."*

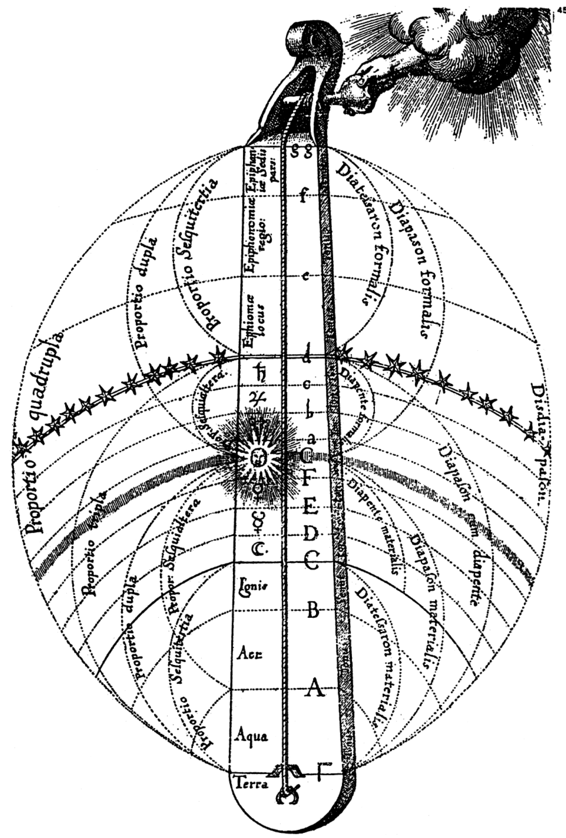
"In the fifth hebdomad [ages 28-35], thanks to the manifestation of the harmonic 35, all increase as regards strength is checked, and after these years it is no longer possible for people to become stronger than they are. ... Finally, when the principle of the decade is blended with

that of the hebdomad and ten times seven is reached, then man should be released from all tasks and dedicated to the enjoyment of happiness, as they say."

45 *The Divine Monochord*

The three realms with their divisions are set out along a monochord. To the immediate left of the string Fludd specifies the members of each realm (giving to the empyrean hierarchy the Greek names of *Epiphaniae*=apparitions, *Epiphonomiae*=voices, and *Ephiomae*=acclamations). To each is assigned a note of the scale, from low G for the Earth (the Greek letter *Gamma*) up through two octaves to gg for the highest division of the empyrean. Applying the proportions of the last diagram, we find that they work perfectly: e.g. the *Proportio dupla* (2:1) from the Earth to the Sun becomes the octave interval from *Gamma* to G. On the right are the Greek names of the musical intervals corresponding to each proportion: *Disdiapason*=double octave=4:1; *Diapason*=octave=2:1; *Diapente*=fifth=3:2; *Diatessaron*=fourth=4:3.

There is an error in the 'Diapente materialis': it should join the Sun's G to the C of fire, as should the corresponding *proportio sesquialtera*. And in order for the tones and semitones to be correct (to the right of the string), we have to imagine the Fs as sharp.
UCH I, a, p. 90



The Cosmic Monochord
after Fludd/Godwin

Figure 16 - The Cosmic Monochord.

Additional Methods of Constructing the Ennagon, Decagon and Dodecagon

The Ennagon, figure 17a - Given interval AB and Vesica Piscis ABCD, with the center of the Vesica at E. Add interval AE to D to find point F, the center of the circumscribing circle. Extend the Vesica circles to find the next two sides of the Ennagon. This is a more elegant version of **figure 10g**.

The Decagon, figure 17b - Given interval AB and Vesica Piscis ABCD, with the center of the Vesica at E. Lay off interval AB on the center line CE from point E to find F. Draw line FB. With center at F and radius AE find points G and H. With center at H and radius BH find point J on the extension of AB. With center at A and radiius AG draw arc to find point K on the extension of center line CD. K is the center of the circumscribing circle.

The Dodecagon, figure 17c - Given interval AB and Vesica Piscis ABCD, with the center of the Vesica at E. Lay off interval AB on the Vesica centerline CD to find point E. E is the center of the circumacribing circle. This is a more elegant version of **figure 10j**.

The Dodecahedron, figure 17d - Given a circle with center at A, radius AB. Construct Vesica Piscis ABCD. Find point E where the circle circumference intersects the Vesica center line AD. EB and EC are sides of the dodecahedron. Critchlow uses the interval AB as a radius to find the vertex's of the dodecahedron.

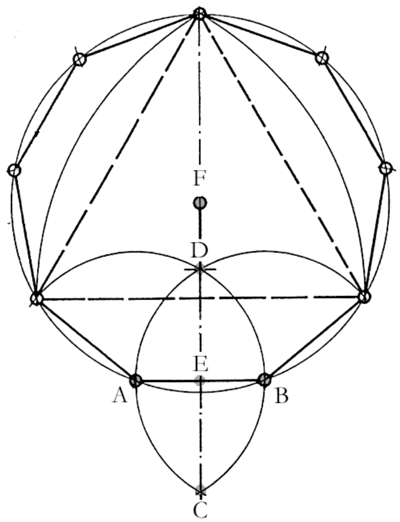


Figure 17a - Ennagon

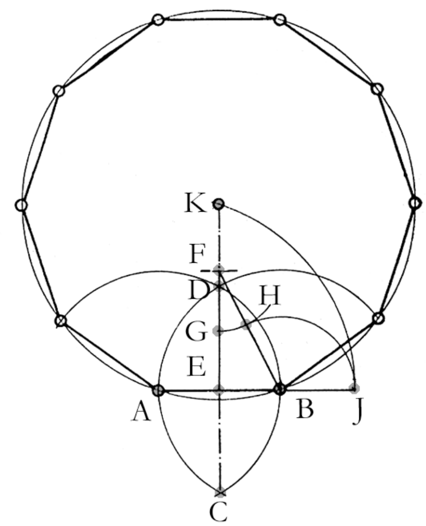


Figure 17b - Decagon

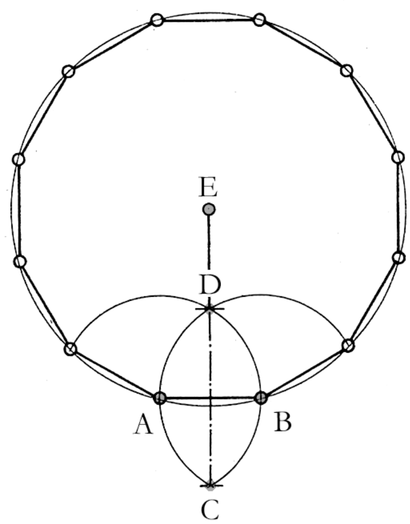


Figure 17c - Dodecagon

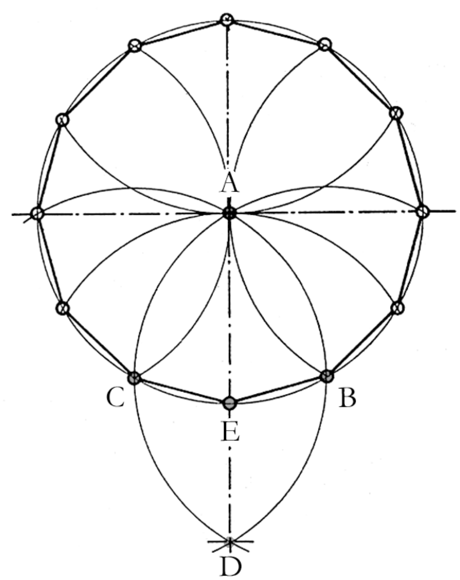


Figure 17d - Dodecagon

Figure 17

There are of course more topics treated in the ancient literature but many are quite obscure especially those related to medical treatment and concepts of what we might call 'human nature'. For example, seven is said to govern the course of fever but its correlation would require lengthy explanation of ancient medical concepts, etc. For the moment, enough said.

Session 4

4.1 Find the center of a given circular arc. 20

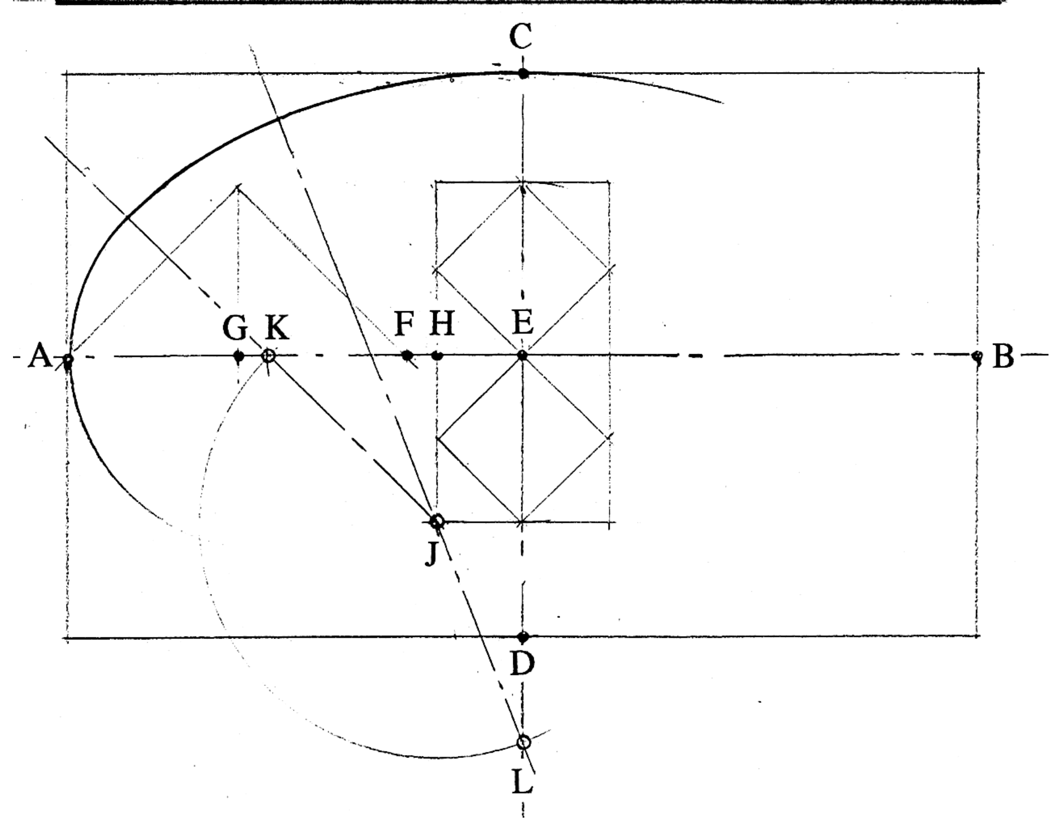
Euclid III.1, III.3 - reverse, III.25, IV.5, Serlio p 20. French, 5.21. Greenberg, p 27

Our construction uses a combination of Serlio's formulation of this idea, with its illustration of what looks like a broken plate. But Serlio and Euclid mark only three points on the arc, not four as we do., for the sake of clarity of demonstration and in emulation of Serlio's practical illustration.

The closest formulation in Euclid is III.25 - 'Given a segment of a circle, to describe the complete circle of which it is a segment.' The proof involves creating two isocilies triangles within the arc. III.1, 'to find the center of a given circle', is also similar.

The next several constructions concern the ellipse. This form dominated the Baroque architectural imagination, exemplified in the work of Borrominni and Berninni. In **figure 18** we provide a constructiion of an eight centered ellipse.

Construct an ellipse given the major & minor axis - 8-center method.



1. Given major axis AB and minor axis CD with center at E.
2. Subtract CD from AB to find point F. Divide AF to find G; AG = AF/2
3. Construct a double square centered on E; EH = AG/2 and HJ = AG.
4. Locate point K on AB; KH = HJ.
5. With center at J and radius KJ draw arc to intersect CD at L.
6. J, K and L are three of the ellipse's centers. The first segment has radius AK, the second has radius LC. Connect the segments with an arc centered on J.
7. Repeat steps 1 - 5 to find the other five centers and draw the arcs in the other quadrants to complete the ellipse.

Figure 18

4.2 Construct an ellipse given the major & minor axis - 4-center method. 21

Not in Euclid. Not in Serlio. French, 5.41. Greenberg, p 45.

4.3 Construct an ellipse given the major & minor axis - perpendicular method. 22
Not in Euclid. Serlio, p 22. French 5.38. Greenberg p 42
See Serlio's problem of the vessel.

4.4 Construct an ellipse given the major & minor axis - diagonal method. 23
Not in Euclid or Serlio. French, 5.37. Greenberg, p 42.

Session 5

Here we begin considering the geometry underlying architectural shapes.

5.1 Construct a cyma recta profile. 24
Benjamin pl. IX, fig. I is cyma recta; fig. E and F are cyma reversa; he gives them as arithmetic, not geometric constructions. Greenberg p, 71.
Vesica construction, an example of 'Ad Triangulum'
Cyma reversa
Unevenly divided cyma

5.2 Construct a 2-centered torus profile. 25
Greenberg p, 66
could be reversed for upward movement
ellipse variant
the scotia

5.3 Construct a quirked ovolo profile - 2-centered method 26
Benjamin pl. X, figs, A, B, and C. Greenberg p, 70.

5.4 Construct a quirked ovolo profile - hyperbolic method 27
Benjamin pl. XI, figs. 1 through 6. Greenberg, p 70.
the circle, elipse, parabola and hyperbola are also known as conic sections, see **figure19**.

5.5 Construct the raking profile of a cyma recta moulding 28
Euclid VI.9-10 'to divide a line into any number of parts'. Serlio, p 17. Benjamin pl. VI, fig. 3
transformational geometry, an example of geometrical algebra

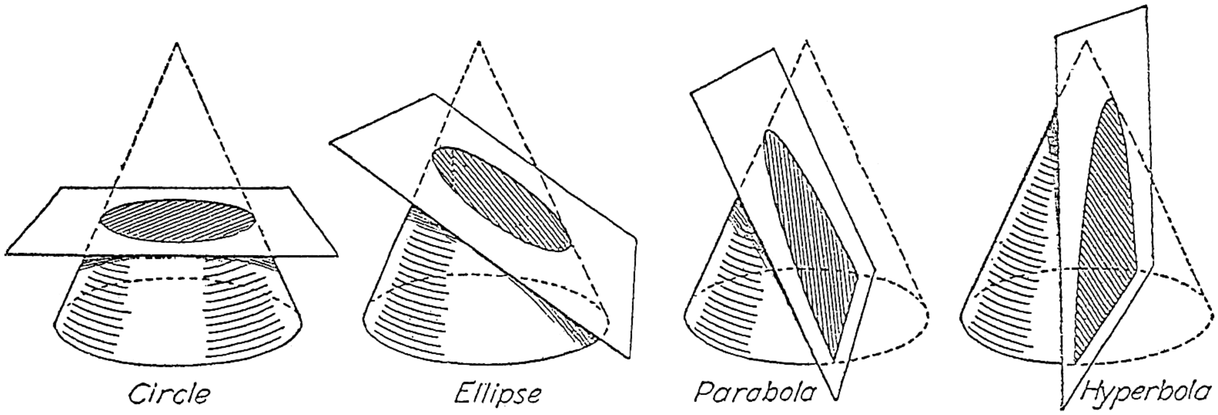
Session 6

6.1 Construct a spiral on an equilateral triangle 29
Greenberg, p 65

6.2 Construct a spiral on a square 30
Greenberg, p 65
this method can be used on any regular polygon.

6.3 Construct a spiral on a golden section rectangle 31
Not in Euclid. Not in Serlio. Huntley, 'The Divine Proportion', p101, fig 7.6; see also his chapter XIII, on spirals.

The Conic sections



The curved forms known as the circle, the ellipse, the parabola and the hyperbola may all be defined by various relations of a plane and a cone. Assuming that the axis of the cone is perpendicular to its base, all planes intersecting the cone parallel to the base will be circles. If the plane is not parallel to the base the curve of their intersectionis an ellipse. If the plane is parallel to the side of the cone the curve defining the intersection is the parabola. If the plane is perpendicular to the base and parallel to the axis of the cone the curve formed is the hyperbola. Each of these curves plays a significant role in architectural planning and detailing

AFTER FRENCH FIG 5.42 FIG. 13A

Figure 19 a

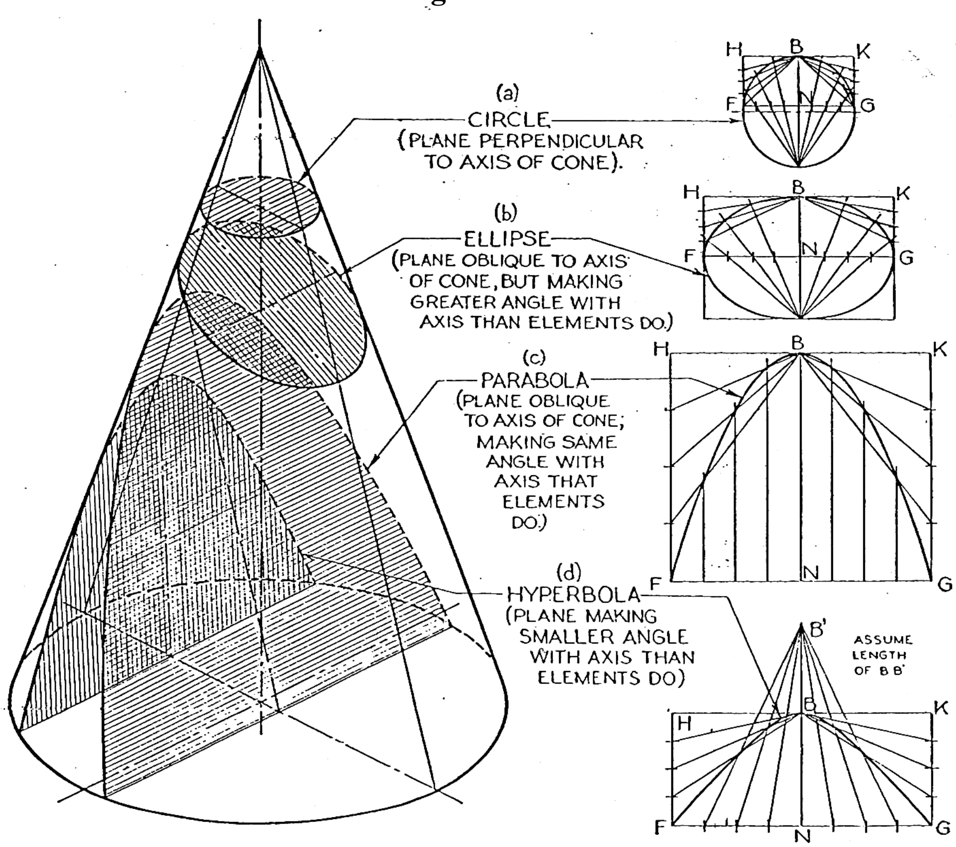
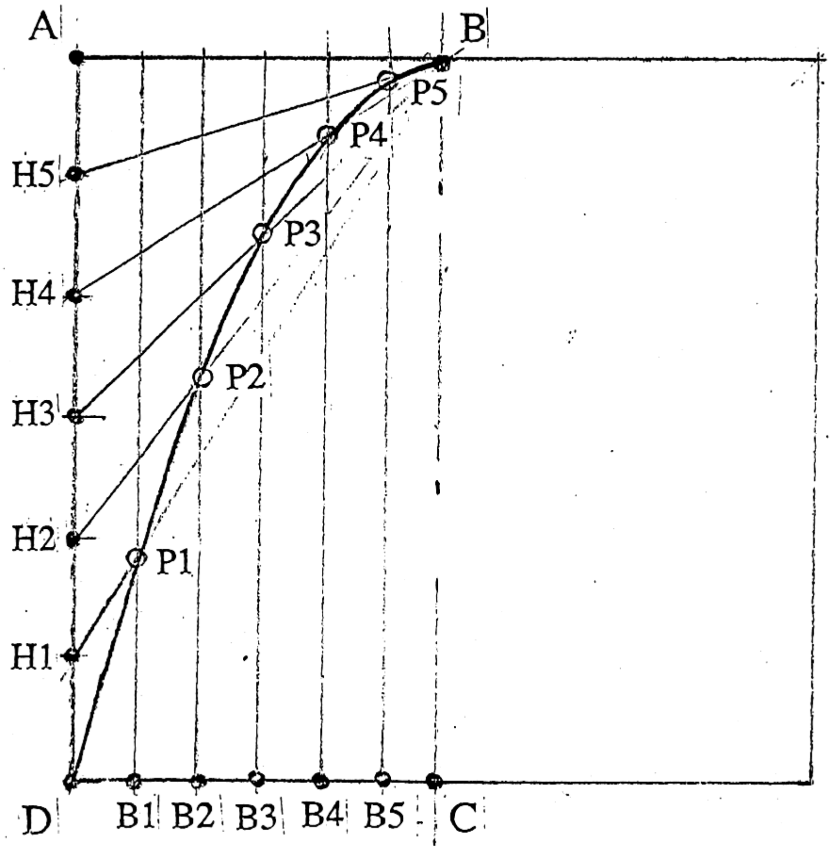


FIG. 152 Conic Sections.

Figure 19 b

Construct a parabola given its altitude and base.



1. Given altitude AD and base. Divide the base to find AB.

2. Divide AB and AD into the same number of parts, H1 thru 5 and B1 thru 5.

3. Project H1 thru H5 to point B and draw lines thru points B1 thru B5 parallel to AB.

4. Plot points of intersection of lines H1 and B1, repeat to find points P1 thru P5. These plus points B and D give the parabolic curve.

Figure 19 c

6.4 Construct the eye of the Ionic volute - Vignola's method 32

Vitruvius, book III, Chapter 5, Sect 5-8. Alberti, Bk 7, ch 8. Not in Serlio book I. Normond plate 49.. Palladio bk I, ch XVI, pl XIX.

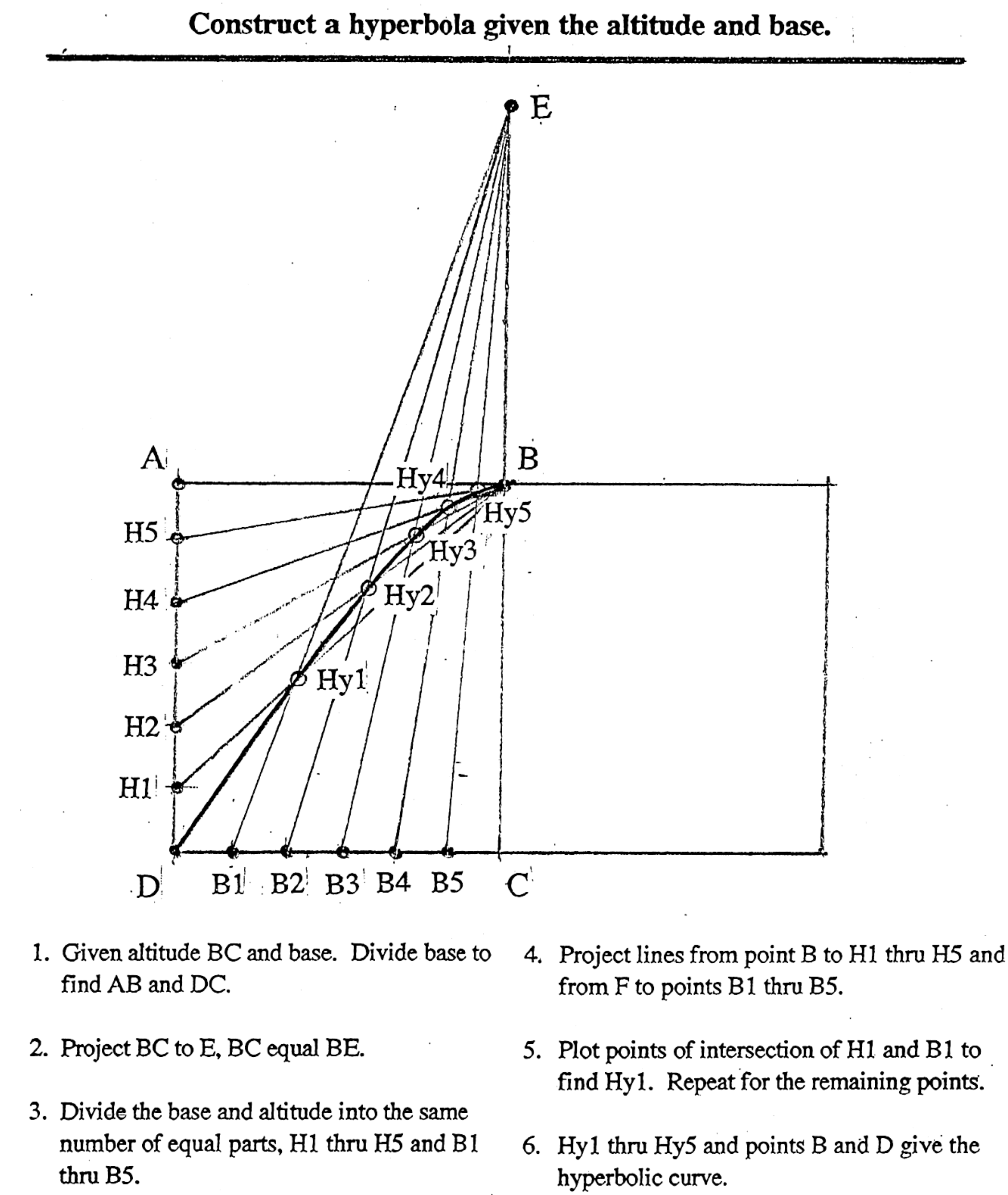


Figure 19 d

6.5 Construct the eye of the Ionic Volute - Goldman's method 33

Not in Euclid, Vitruvius, Alberti, Serlio, Palladio, Vignola or Gibbs. First published by Chambers, 1759, page 52-53 of the Dover edition of 2003. Normond's Parallel, plate 50.

6.6 Construct the Ionic volute - Vignola's method 34

Vitruvius, bk III.5, 5-8. Alberti, Bk 7, ch 8. Palladio bk I, ch XVI, pl XIX. Vignola, plate 20. Serlio Bk. IV, p 322 Normond, plate 49.

6.7 Construct a modillion volute - Gibbs Method

This method is not given by any other canonical author. Gibbs, Rules for Drawing …, 1732, Plate LXIV. **Figure 20**.

Construct a modillion volute - Gibbs method

1. Given square ABCD, divided into 8 x 8 units, **figure 20a**. It will be 8 units high and 7 units wide.

2. The eye will be 1 unit in diameter, located at 4.5 units vertically and 4 units horizontally.

3. The eye is internally structured as in construction 6.6.

4. For this volute we will use only 8 arcs, labeled 'a' through 'h', rather than 12 as in construction 6.6.

5. Centers a, b, c, and d are located as in construction 6.6 but centers e, f, g, and h are located at the quarter points of the center lines of the square, rather than the third points as in 6.6.

6. Draw arc hj. The rosette is centered at K with radius Kj.

7. As interval 'ae' within the eye gets smaller the spiral winds more tightly; as 'ae' gets larger the spiral winds more slowly, affecting the size of the rose.

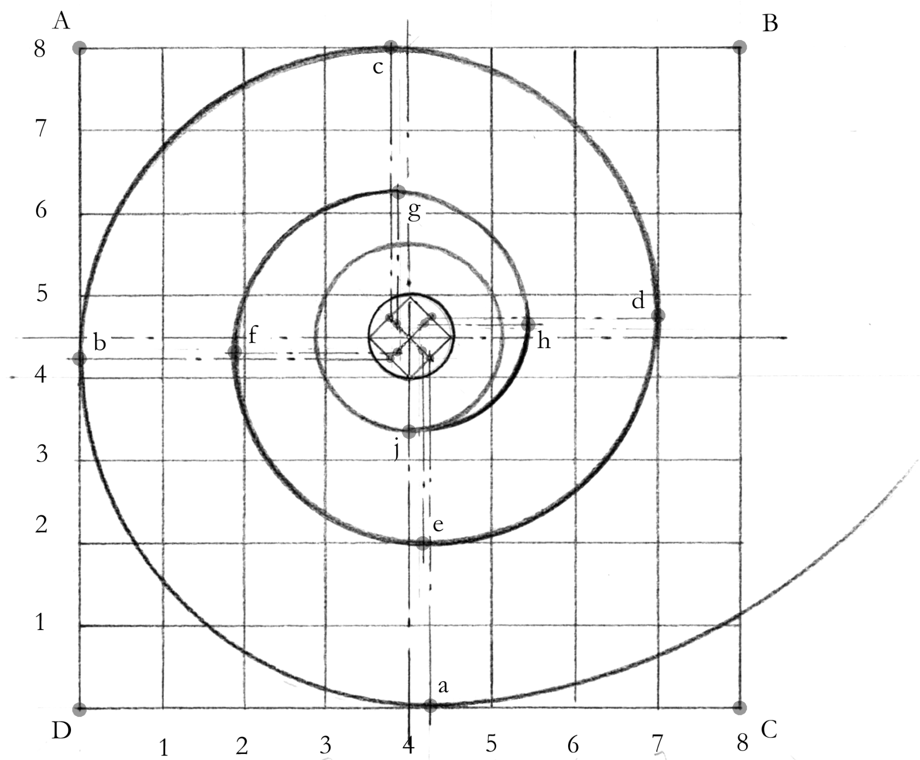


Figure 20a - The Volute

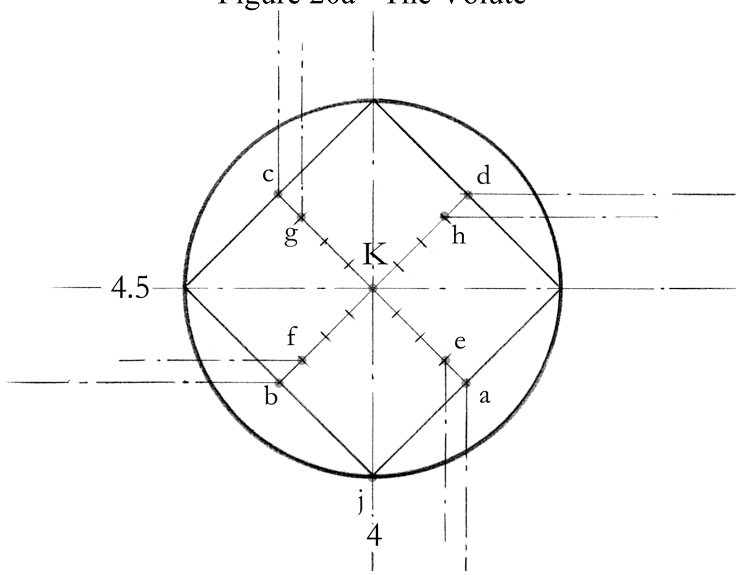


Figure 20b - The Eye

Figure 20

6.8 Construct a decorative volute - d'Aviler's method

This method is given in Normand, plate 50. The modified method shown here, **figure 21**, is given by Benjamin, plate XLIV, with reference to stair rails.

In **figure 20** we begin with the overall size of the spiral and the size of the rosette is determined afterward. In this figure we start with a center to edge interval and the size of the rosette. The advantage of this method is that we can have a spiral of any number of turns and a rosette of any size.

Construct a Decorative Volute - d'Aviler's method

1. Given interval AB. Set the radius of the Rosette, AC. In this example the radius is a little less that 1/4 of AB, similar to figure 20 where it is a little more than 1/4.

2. Divide circle AC into 8 parts and extend the dividers to length AB.

3. With center at B and radius BC draw arc CD. Determine the number of turns you want the spiral to take and multiply by 8, in this example 8 x 2 = 16. Divide the arc into 16 equal intervals.

4. At each interval draw lines, parallel to BD, to BC, *1* through *15* [only *1* - *12* are labeled in the figure for clarity's sake].

5. With center at A and radius A*1* mark point 1 on divider 1A5. With center at 1 and radius AB find point 1' on divider 2A6. With center at 1' and radius 1',B draw arc B,1. Note that the tangent point to BD will be slightly to the right of B.

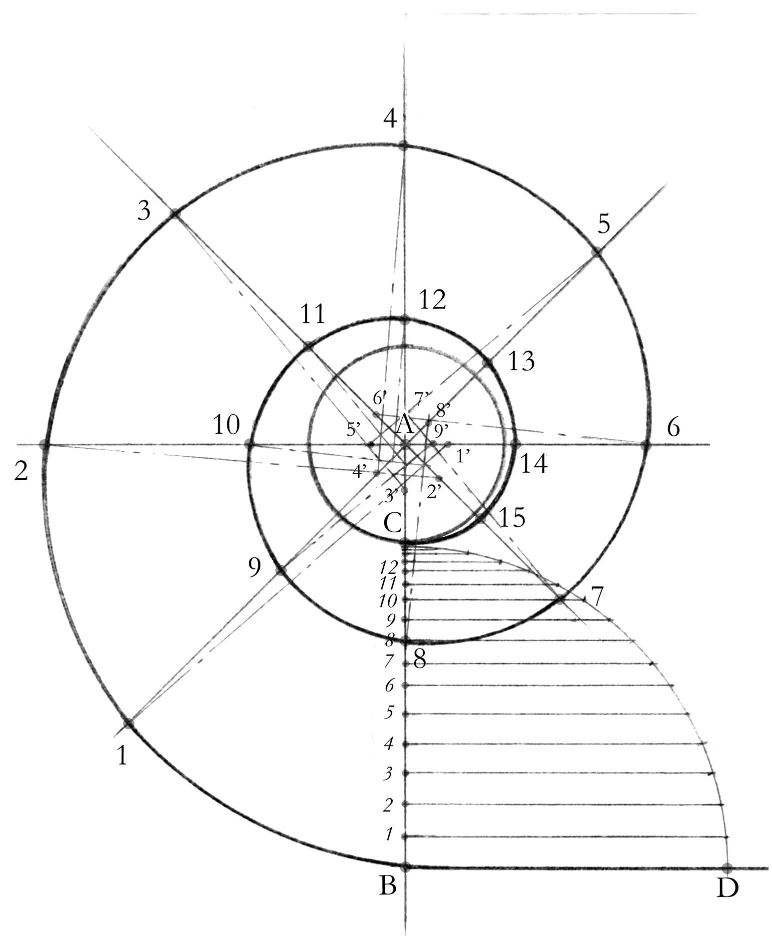


Figure 21

6. With center at A and radius A*2* find point 2 on divider 2A6. With center at 2 and radius A*1* find point 2' on divider 3A7. With center at 2', radius 1,2' [= A*1*] draw arc 1,2.

7. With center at A, radius A*3* find point 3 on divider 3A7. With center at 3, radius A*2* find 3' on divider 4AB. With center at 3', radius 2,3' [= A*3*] draw arc 2,3.

8. Repeat this process to find points 4 through 15 and their respective arcs. In dragting this figure it mey be easier to find points 2 through 15 first, then find centers 1' through 15' drawing the arcs as you go [only 1' through 9' are shown on the illustration for clarity's sake].

6.9 Find the height of a room given its length & width - Palladio's method .

Palladio, Bk I, chapter XXIII. See **figure 22**.

Palladio uses the three means to find the heights of rooms, recalling Plato's use of these as part of the cosmic construction in the Timaeus.

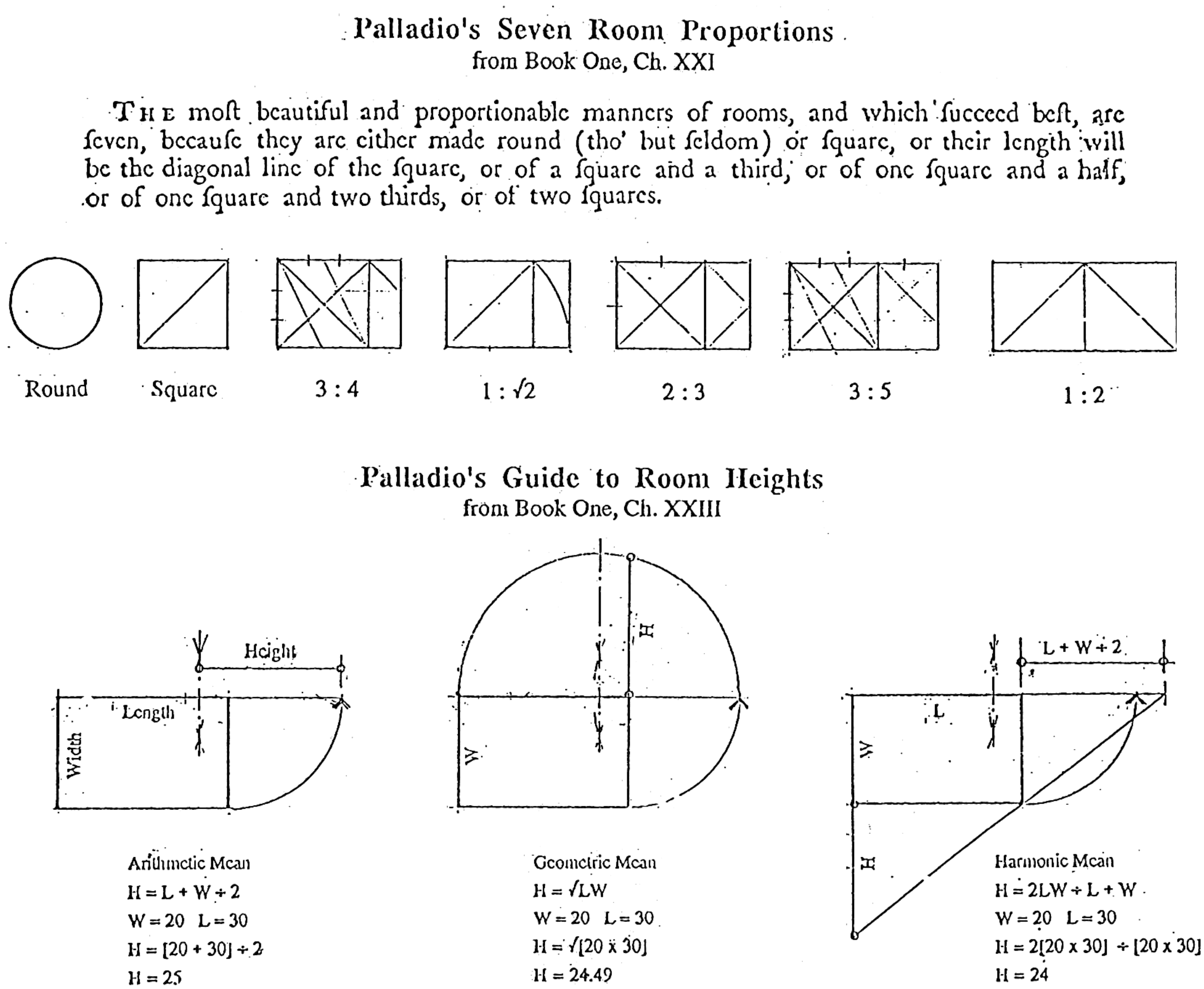


Figure 22

6.10 Construct an Arch with Voussoirs

This construction has no specific references in the architectural canon though the process is probably described in various masonry craft texts. The method used here is an approximate one, used to align the voussoirs with horizontal stone coursing. There are many varients of this construction.

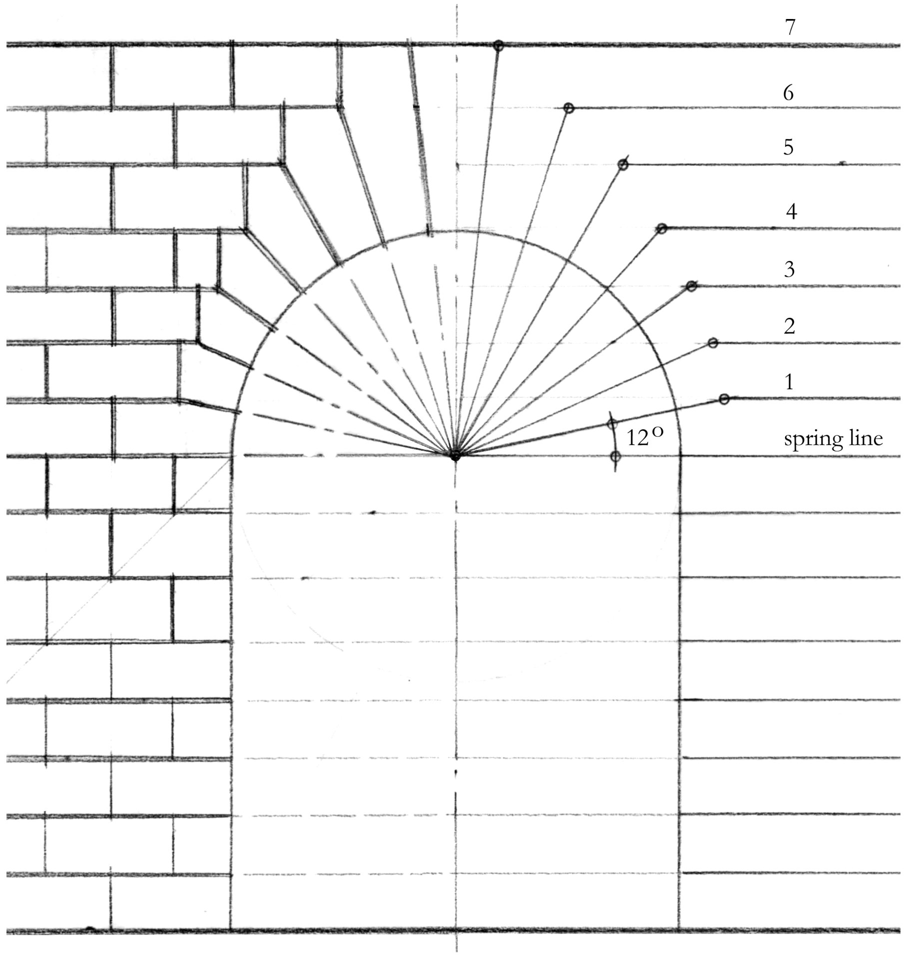


Figure 23

1. Set the number of courses above the spring point.

2. Multiply the number of courses by 2 and add 1 for the keystone.
In this case 7 x 2 = 14, 14 + 1 = 15.

3. Determine the angular division: In this case $180^o \div 15 = 12^o$.
Draw radaii from the center at 12^o intervals; extend to where they intersect the coursing. Mark a vertical joint at each point.

4. Alternate: Take 3/4 to 7/8 of the course height; lay off this interval on the arch and adjust to create equal intervals.

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End of Commentary